PASSIVE MODIFICATION AND ACTIVE VIBRATION CONTROL BY THE RECEPTANCE METHOD

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy

by

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October 2007

To my father

ABSTRACT

In this thesis the receptance method is developed and applied to passive modification and active vibration control. The methodology is well known in passive modification, however requires and is further developed for the inverse problem in active vibration control. The receptance matrix $\mathbf{H}(s)$ is defined by inverting the dynamic stiffness matrix, however in practice it may be obtained from the measured frequency response function $\mathbf{H}(\mathrm{i}\omega)$. Therefore there would be no requirement to know or evaluate the system matrices **M**, **C**, **K** typically obtained from finite elements.

There are numerous disadvantages to conventional state-space analysis using the system matrices **M**, **C**, **K**. Firstly, the many different forms of damping in structures cannot all be represented by the standard form of second order matrix differential equations. FE models normally neglect damping or assume an *ad hoc* Rayleigh (proportional) damping. In active control the damping model is very important in the eigenvalue assignment problem and plays an important role in the closed-loop system stability. Secondly, FE models used in design can be very large therefore computationally expensive and require model reduction, truncation or other approximations which can degrade the performance of the controller. Finally the controller should be insensitive to the ill-defined FE parameters such as joints and boundary conditions. These uncertainties in the system parameters may result in lack of robustness in the closed-loop system.

These disadvantages do not apply to the inverse problems using the receptance method. Since the method uses receptances rather than dynamic stiffness, the system equations are complete with just a small number of

states. This means that there is no need for model reduction or the estimator to estimate the unmeasured states using an observer.

The receptance method is applied to the passive modification of a Lynx Mark 7 helicopter tailcone. The modified receptances are obtained from the initial receptances and the known modification.

The receptance method is also applied to problems in active vibration control using state feedback and output feedback. The characteristic equations for the assignment of poles, zeros and simultaneous assignment of poles and zeros are developed. The methodology is demonstrated by numerical examples as well as experimental work. It is shown that the sensitivities of the poles may also be assigned using the measured receptances.

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ACKNOWLEDGMENTS

I would like to express sincere gratitude to my supervisor, Professor John E. Mottershead, for his invaluable support, encouragement, supervision and useful suggestions throughout this research work. His support and continuous guidance enabled me to complete my work successfully. I am also highly thankful to Dr Simon James, for his valuable help throughout my experimental work.

I would also like to thank the U.K. Government for the award of the Overseas Research Students Award Scheme (ORSAS) and the University of Liverpool for the award of a scholarship, which has supported me during my research.

My thanks to my colleagues and academic members of Dynamics and Control Research Group in the University of Liverpool and also in Iran University of Science and Technology for their kind suggestions and support.

And finally, thanks to my father and my friends especially Y.W.Chai, Pheonix Yang, Kate Whitfield, who endured this long process with me, always offering support and love.

NOMENCLATURE

$\mathbf{A}$	System matrix
$\mathbf{B}$	Control distribution matrix
$\mathbf{b}$	Single-input control distribution matrix
$\mathbf{C}$	Damping matrix
$\mathbf{D}$	Measurement distribution matrix
d	Denominator of the receptance transfer function $\mathbf{H}(s)$
$\mathbf{E}$	Acceleration feedback control gain matrix
$\mathbf{e}_i$	Unit vector at coordinate i
$\mathbf{F}$	Velocity feedback control gain matrix
f_i	Term in control gain matrix $\mathbf{F}$
$\mathbf{G}$	Displacement feedback control gain matrix
g_i	Term in control gain matrix $\mathbf{G}$
$\mathbf{H}$	Receptance matrix
$\widetilde{\mathbf{H}}$	Modified receptance matrix
h_{pq}	pqth term of receptance matrix
$\mathbf{I}$	Identity matrix
$\mathbf{K}$	Stiffness matrix
$\mathbf{M}$	Mass matrix
$\mathbf{N}$	Numerator of the receptance transfer function $\mathbf{H}(s)$

S_{ji}^{f} Sensitivity of jth eigenvalue with respect to feedback gain f_i

S_{ji}^{g} Sensitivity of jth eigenvalue with respect to feedback gain g_i

s Frequency in the complex plane

$\mathbf{u}$ Control force

$\mathbf{V}$ Matrix of open-loop eigenvectors

$\mathbf{v}$ Eigenvector associated with open-loop eigenvalue σ

$\mathbf{x}$ Displacement in physical coordinate

$\mathbf{y}$ Vector of outputs

$\mathbf{Z}$ Dynamic stiffness matrix

$\Delta\mathbf{Z}$ Dynamic stiffness of modification

Δ Time delay

$\mathbf{\Lambda}$ Diagonal matrix of closed-loop poles

λ Closed-loop poles

μ Closed-loop zeros

ω The imaginary part of the frequency in rad/sec

$\mathbf{\Sigma}$ Diagonal matrix of open-loop poles

σ Open-loop poles

$\underline{\sigma}$ Minimum singular value

ζ Damping ratio

Chapter 1

INTRODUCTION

1.0 Introduction

In this thesis the receptance method is developed and applied in two directions (i) passive structural modification and (ii) active vibration suppression by pole-zero assignment. The receptance method in passive structural modification may be traced back to the work by Duncan [D15] who determined the dynamic behaviour of a modified system from the receptances of the initial system and the known modification. The theory was further developed for the inverse problem of eigenvalue assignment and used for vibration suppression in structural dynamics. The receptance method is well known in passive structural modification, but was introduced for the first time in active vibration control by Ram and Mottershead [R6]. The objective of the present work is to develop the receptance method for the inverse problem of eigenvalue assignment in both passive structural modification and active vibration control.

Vibration suppression by the receptance method has wide applications in many industries. For example in the automotive industry vibration and noise control is of extreme importance to passenger comfort. In helicopters the vibration isolation of the pilot's seat is one of the main problems. In manufacturing industry the vibration absorption of machine tools lead to more reliable and higher quality products. The objectives of vibration suppression in these industries are difficult to achieve at the design stage by other techniques such finite elements. Industrial finite element models of structures such as cars or helicopters are usually very large, sometimes millions of degrees of freedom, and are therefore computationally expensive and require model reduction, truncation or other approximations. They may not be accurate particularly in the representation of damping and can be ill-defined at the joints and boundary conditions. Therefore, by adopting the receptance method one can avoid the difficulties associated with the finite element models. The receptance marix $\mathbf{H}(s)$ may in theory be determined by inverting the dynamic stiffness matrix, but in practice would be obtained from the measured frequency response function $\mathbf{H}(\mathrm{i}\omega)$, which relates the translational/rotational displacement response of the structure to the input force/moment excitation.

Passive modification may take the form of masses, springs and dampers added to the structure. The receptance method may be applied to determine the modified system receptances from the measurements of the initial receptance matrix. Physical modifications such as beams, plates or overhanging masses require the measurement of rotational receptances as well as the translational ones. Mottershead *et al.* [M13] introduced the T-block attachment to measure the rotational receptances of a modified structure by an added beam [K7]. However, the full receptance matrix cannot be obtained using the T-block attachment. For example the cross-rotational terms are unmeasured. The use of an X-block attachment is proposed in this research, which enables us to measure the full receptance matrix. The structural modification theory based on the receptance method is applied to a Lynx Mark 7 helicopter tail-cone. The passive modification in the form of a large overhanging mass is considered, representing the mass of the tail-rotor gear box and hub. The flexibility of the X-block as well as the measured spectral densities is included in the formulation of H_1 and H_2 estimators to determine the receptance matrix at the connection point of the X-block to the tail-cone. Moving the X-block into three different positions enables the estimation of the full 6×6 receptance matrix. Structural modification of the helicopter tailcone demonstrates the application of the receptance method in the forward structural modification problem [M15].

Inverse problems for the assignment of poles and zeros have been studied for many years using passive modifications such as point masses, springs, beams and plates. The poles and zeros (resonances and antiresonances) of the system, which are defined in the complex plane, may be assigned to desired locations by structural modification. As an example, the natural frequencies of the structure may be shifted to desired frequencies to avoid the large amplitude vibrations close to resonances. The main advantage of passive modification over active vibration control is that the system is guaranteed to be stable. However, there are considerable disadvantages; the form of modification required for the pole or zero assignment may not be realizable in practice; the rotational degrees of freedom may be important and require an expensive procedure to measure them as demonstrated in the structural modification of the helicopter tail-cone; and also the number of poles to be assigned cannot be greater than the rank of the modification.

These disadvantages do not apply to eigenvalue assignment by active vibration control. The main issue in active control is the stability of the closed-loop system. In this research eigenvalue assignment by the receptance method for active vibration control is developed based on the work of Ram and Mottershead [R6] using the state feedback. The same methodology used in passive modification is applied to active vibration control. The receptance transfer function $\mathbf{H}(s)$ is determined by inverting the dynamic stiffness matrix, but in practice it would be obtained from the measured frequency response function $\mathbf{H}(i\omega)$ [M16]. The significant advantage of the receptance method is that it does not require evaluating mass, damping and stiffness matrices typically obtained from FE models.

Since the method uses receptances rather than dynamic stiffness, the system equations are complete by measuring the states at sensor locations. This means that there is no need for model reduction or the estimation of the unmeasured states by an observer.

A new theory for the output feedback in active vibration suppression is developed by the receptance method [M16]. The very considerable advantage of the output feedback method over state feedback is that it allows the use of collocated sensors and actuators in multiple-input-multiple-output systems. It is well known that use of the collocated sensors and actuators guarantees the stability of the closed-loop system due to the interlacing pattern of poles and zeros. The assignment of poles and zeros separately and simultaneously are achieved by output feedback. In addition to velocity feedback, for active damping, the displacement feedback is considered, for active stiffness, thereby enabling the assignment of poles and zeros to desired locations in the complex s-plane. Characteristic equations with the unknown feedback gains are formulated for the assignment of poles and zeros.

Experiments are carried out on a T-shaped plate with collocated sensors and actuators for the output feedback control to demonstrate the feasibility of the technique [M16]. The assignment of two pairs of complex conjugate poles corresponding to the first two modes, assignment of zeros and simultaneous assignment of poles and zeros are considered. The open-loop receptance is determined from the measured frequency response function $\mathbf{H}(\mathrm{i}\omega)$ and a rational fraction polynomial is fitted to represent the transfer function of the receptance. The control gains are obtained by solving the nonlinear characteristic equations using the Newton iterative method. The stability robustness of control systems is analysed using the minimum singular values of the return difference transfer function matrix.

It is often desirable not only to place eigenvalues at chosen locations in the complex plane, but also to design controllers that render selected eigenvalues sensitive while others are made insensitive. The sensitivities of the eigenvalues with respect to the control gains are determined from the matrix of measured receptances.

1.1 The scope of thesis

The structure of this thesis is as follows; Chapter two reviews the structural modification and active vibration control literature, mainly concentrating upon inverse problems of pole-zero assignment. Chapter three presents the general theory of the receptance method which can be applied to both passive modification and active vibration control. The theory of rank 1 modification including Vincent Circle theory and general modifications of higher order are described. Chapter four demonstrates the problem of structural modification on the Lynx Mark 7 helicopter tail-cone. A technique to measure the rotational receptances is proposed using the formulation of multiple-input-multiple-output estimator and the finite element model of the attachment. Chapter five describes the partial eigenvalue assignment problem by state feedback control using orthogonality relationships between the mass, damping and stiffness matrices. The two cases of single-input and multiple-input feedback control are considered for the eigenvalue assignment. Chapters six and seven present the receptance method in the assignment of poles and zeros using state feedback control and output feedback control respectively. Chapter eight describes an experimental example of output feedback control on a T-shaped plate using collocated sensors and actuators. Experimental results include the assignment of poles, assignment of zeros and simultaneous assignment of poles and zeros. Chapter nine addresses the problem of assigning sensitivities of the system poles to small changes in the control gains. Finally, Chapter ten presents the principal conclusions of this research and suggestions for future work.

Chapter 2

LITERATURE REVIEW

2.0 Introduction

The purpose of this chapter is to present an overview of the structural modification and active vibration control literature in order to understand the logical development of the subject to the present time. The structural modification literature includes (i) forward structural modification, (ii) inverse problems for pole-zero assignment using the receptance method and (iii) the measurement of rotational receptances.

The active control literature covers (i) eigenstructure assignment, (ii) robust control, (iii) active damping and finally (iv) independent modal space control. The inverse problem of pole-zero assignment is discussed in detail for both structural modification and active vibration control.

2.1 Structural modification

The problem of vibration suppression by means of structural modification has occupied researchers for at least a century [F4], [D15]. There are two approaches for structural modification problems: forward and inverse. The forward modification is to determine the eigenvalues and eigenvectors of a system with known modifications to its mass, stiffness and damping terms. The inverse problem is to determine the modification which will bring about the desired change in the eigenvalues and eigenvectors of a vibrating system. We begin with a brief overview on forward structural modification literature; however a more detailed survey is provided for inverse problems.

2.1.1 *Forward structural modification*

Structural modification is about changing the dynamic stiffness of a mechanical system. The problem of determining the dynamical behaviour of a combined system from two or more sub-systems with known initial receptances and known connection properties was addressed in Duncan's article of 1941 [D15] and Sofrin's paper of 1946 [S6]. Bishop and Johnson [B11] addressed the direct structural modification problems and described the receptance method in great detail. Ram [R4] presented a very useful technique based on the receptances of the initial system to determine the dynamic behaviour of the modified system formed from mass, spring and dashpot modifications. He derived the eigenvalues of damped subsystems with the known modifications using transfer functions and modal data from the separate subsystems. Ewins [E4] described the prediction of an assembled or modified structure from knowledge of the corresponding dynamics of its component parts. However, a problem of ill-conditioning arises with the inversion of the matrix of connection-point receptances [B8],[B9]. The forward structural modification is well understood and the focus nowadays is on inverse structural modification.

2.1.2 *Inverse problems*

The objective of inverse problem studied in this thesis is to determine the modification which assigns the desired dynamical behaviour of a vibrating system. Most inverse structural modification problems focus on assignment of natural frequencies and antiresonances. The natural frequencies of a structure may be shifted to desired locations by adding modification such as point masses, springs, beams or plates so that the system responds in a desirable way. Anti-resonances are as important as the natural frequencies since they are the frequencies at which the vibration disappears to zero, or to low amplitudes when damping is present. The vibration absorber invented by Frahm [F4] in 1909 is the first simple modification designed to assign an antiresonance at a prescribed frequency. The vibration absorber theory is addressed in P.Den Hartog's book [D11] of 1940. Nowadays, more complicated modifications in the form of passive modification and active control are applied for the problem of vibration suppression.

The inverse structural modification problem may be traced back to the work of Weissenburger [W1] in 1968. The method proposed by Weissenburger was receptance modelling for the assignment of a single natural frequency by a unit rank modification. Pomazal and Synder [P1] extended this methodology to the case of damped systems and determined the natural frequencies and mode shapes of the system that had been modified by addition of a unit rank matrix. A general approach for the assignment of natural frequency based on the Rayleigh quotient was described by Dowell [D12].

Receptance modelling was applied for the assignment of anti-resonances in the UK helicopter industry in 1972. Vincent [V2] discovered that when a structure excited at a point q with a constant frequency is modified, for example by the addition of a spring between two coordinates r and t, then the response at another point p traces out a circle when plotted in the complex plane as the spring stiffness is varied from minus to plus infinity. The problem of vibration absorption then reduces to finding the point on the circle closest to the origin of the complex response. Nagy [N1] developed the Vincent circle theory by including the spring-mass absorber. Vibration suppression using Vincent circle analysis was extended by Ghandchi Tehrani *et al.* [G5] which includes the mass, stiffness and damping modification at different coordinates.

Vibration suppression can be achieved by assignment of zeros (antiresonances). Zeros are the frequencies in the complex plane at which the vibration response vanishes. For an undamped system, zeros lie on the imaginary axis of the complex plane. Zeros of the system can be determined mathematically by solving for the eigenvalues of the adjoint system, obtained by deleting a row and a column from the original dynamic stiffness matrix. If the row and the column have the same index, the resulting system matrix is symmetric and therefore the eigenvalues are the zeros of the point receptance. However, when the indices of the deleted row and column are different then the resulting matrix is not symmetric and zeros of the cross receptance may become complex. Mottershead [M8] studied the relationship between the sensitivities of the zeros and the sensitivities of the natural frequencies and mode-shapes of structural systems. Mottershead and Lallement [M9] used this knowledge together with the theory of unit rank modification to assign natural frequencies and anti-resonances at the same value thereby creating a vibration node. When a pole and a point-receptance zero coincide at the same eigenvalue a vibration node occurs at that coordinate. Assignment of vibration nodes of normal modes by the cancellations involving either repeated poles or repeated zeros was addressed by Mottershead *et al.* [M10].

An interesting aspect of receptance method is to define the modified receptances from the measured original receptances. This technique was advocated by Rade and Lallement [R1] who established the anti-resonances constraints. The artificial boundary condition proposed by Gordis [G8] allowed the modified receptances to be defined from the measured receptances without physically applying the modification and altering the boundary condition. The receptance matrix of the constrained, generally damped, system was established in terms of the receptance of the unconstrained system and the coefficients of the linear constraint equations [G9].

Ram and Braun [B12] considered the problem of structural modification for a truncated modal data as an optimised inverse eigenvalue problem. The objective of the optimisation was to find the modification which best approximated the desired response. Butcher and Braun [B14] extracted the left eigenvectors from the measured receptance data to assign the mode shapes. It was concluded that if the left truncated eigenvector is included in the modification procedure an exact solution without any truncation error may be obtained. However, the left eigenvectors extracted from modal test data are very sensitive to measurement noise and regularisation methods were suggested in order to reduce the sensitivity of the problem to measurement noise.

Mottershead [M11] considered the assignment of zeros using measured receptances. The assignment of anti-resonances was accomplished for the first time, without the use of a classical vibration absorber, in a physical experiment by adding masses to a beam. The advantage of this technique is that a zero assignment can be achieved for both point and cross receptances and the modification can be applied to a different coordinate, whereas an absorber must be attached at the coordinate where the point receptance zero will be assigned. An inverse method for the assignment of natural frequencies and vibration nodes by the addition of grounded springs and concentrated masses was presented [M12]. This method was based entirely on the measured receptances at the coordinates of the nodes and the modification. The modification parameters were derived from an analysis of the null space of a matrix containing the measured receptances.

Ram and Elhay [R2] designed a multi degree of freedom dynamic absorber for the absorption of several frequencies. Cha and Pierre [C4] considered passively imposing a node to a linear system in free vibration using an oscillator chain. The oscillator parameters were selected from an inverse eigenvalue problem to place node anywhere along the structure and for any normal mode. The idea was based on the classical absorber introduced by Frahm [F4]. For such an absorber if a natural frequency of the combined system coincides with the natural frequency of the grounded absorber then the vibration node will occur at the coordinate where the absorber is attached to the system at that mode. To impose vibration nodes for n modes simultaneously a chain of n oscillators are required whose grounded natural frequencies are equal to the natural frequencies of the combined system for the n modes at the desired coordinate.

Singh and Ram [S4] proposed both passive and active-control methods for vibration absorption problems. Necessary and sufficient conditions for the stability of the feedback control implementing a single sensor and actuator was discussed. Sivan and Ram [S5] extracted a set of physical solutions that could be realised by physical modifications to the system from a family of mathematical solutions. Kyprianou and Mottershead *et al.* [K6] considered the problem of assigning natural frequencies using an added mass connected to one or more springs. The modification was determined from the solution of a polynomial for the assignment of a single natural frequency. It was shown that for the assignment of more than one frequency, a system of non-linear multivariate polynomial equations should be solved and an added coordinate should be included in the modification. The same authors modified a Γ-shaped frame by an added beam, thereby turning it into a portal frame, while assigning certain natural frequencies and antiresonances [K7]. The addition of a beam requires the measurement of rotational receptances at the modification coordinates, the subject matter of the next section.

2.1.3 Rotational receptances

The measurement of rotational receptances is essential in many structural modification problems, at the connection points where the modification takes place. For practical modifications such as added beams or large masses the rotational coordinates are required to connect the modification to the initial structure. Therefore, the measurement of translational receptances only may not be sufficient to accomplish the structural modification. The problem of measuring the rotational receptances can be separated into the two sub-problems of (i) measuring the rotational displacement (ii) exciting the structure with a moment and measuring it. The first sub-problem is the easiest and many papers have been focused on this aspect. Many attempts have been made to apply a pure moment, which is extremely difficult to implement in practice. An alternative way is to apply a force, which simultaneously imparts a moment and from the measured linear acceleration responses determine a matrix of receptances using a multi-input-multi-output estimator. Many techniques based on mass-additive techniques, estimation techniques, laser Doppler vibrometer and rigid block attachment have been suggested to estimate the rotational receptances, which are described in detail in the references of [M13]. Among the proposed techniques, the most common one, known as rigid block attachment, is described in the present work.

A rigid attachment such as the T-block was considered to measure the rotational receptances [E1], [E2], [E3]. The force is applied to the T-block attachment, which imparts both a force and a moment to the structure at the connection point. The matrix of receptances can be determined from the measured translational receptances, a coordinate transformation matrix and the mass matrix of the attachment. A similar approach using a rigid 'L'-shaped attachment was presented [C5], [Q1]. Maia *et al.* [M1], [M2] developed a mass uncoupling method (MUM) or a mass cancellation procedure, so that the estimated receptances were free of the influence of the added mass of the attachment. The point receptances with and without the T-block attachment were measured and the mass and inertia of the added T-block were removed from the estimated receptances. Silva *et al.*

[S3] proposed a method to extract the rotation-moment receptance from a single column of the receptance matrix corresponding to an applied force. The mass and inertia of the T-block were eliminated using the MUM technique. Yasuda *et al.* [Y1], Kanda *et al.* [K2] and Brown *et al.* [B13] also used rigid attachments; however, they concentrated on measuring the rotational displacements and did not extend their method to the determination of rotational receptances. All methods above using rigid attachments are ill-conditioned for two reasons; (1) rotational motion is obtained from the small difference between translations measured by accelerometers on the rigid attachment. (2) the excitation points on the rigid attachment are close and therefore the receptances on the attachment excited from close points are very similar. The ill-conditioning problems can be alleviated by introducing a flexible attachment to the structure and including the dynamic stiffness of the attachment in the formulation of the estimator. Mottershead *et al.* [M13] developed a technique which includes the flexibility of the attachment in the formulation of multi-input-multi-output estimator. In this approach, the finite element model of the T-block attachment with the measurements at accessible points on the T-block at the location of force sensors and accelerometers are inserted in the formulation of H_1 estimator. However, the full receptance matrix cannot be obtained using the T-block attachment, such as the cross-rotational terms that are unmeasured. The use of an X-block attachment is proposed [M15], which enables us to measure the full receptance matrix. The measurement of rotational receptances is difficult and requires specialist skills. This is one of the disadvantages of passive structural modifications over active control techniques. Active vibration control does not require the measurement of rotational receptances since the modification is not applied physically to the structure.

2.2 Active vibration control

Active vibration control involves the design a controller that prescribes the closed-loop system response. Active vibration suppression is the main objective in structural dynamics and is achieved by strategies such as (i) eigenstructure assignment, (ii) robust control, (iii) active damping and (iv) independent modal space control. We begin with the inverse problem of eigenstructure assignment.

2.2.1 Eigenstructure assignment

The inverse problem of eigenvalue assignment has received considerable attention from the active control and vibrations communities. The assignment of poles and zeros has many potential applications in structural dynamics. For instance, large-amplitude vibrations close to resonances can be avoided by moving the poles of the system to the desired locations. The problem of relocating the poles of the system is called *eigenvalue assignment.* The inverse problem of assigning poles and zeros by adding structural elements such as masses, springs, beams and plates was discussed in previous sections. The advantage of passive modification is that the modified system is guaranteed to be stable. However, there are considerable disadvantages such as 1) the form of the modification that can be realised in practice (symmetry, positive-definiteness, reciprocity, bandedness of the matrix) is restrictive, 2) rotational receptances are very difficult to measure and require high levels of specialist expertise, and 3) the rank of the modification should be at least equal to the number of eigenvalues to be assigned. These disadvantages do not apply to eigenvalue assignment by active control.

If we consider the eigenvalue assignment problem for a time-invariant linear multivariable system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2.1}$$

where $\mathbf{x} \in \Re^{n \times 1}$ is the vector of states, $\mathbf{u} \in \Re^{m \times 1}$ is the control input, $\mathbf{A} \in \Re^{n \times n}$ is the system matrix and $\mathbf{B} \in \Re^{n \times m}$ is the control distribution matrix of full rank. The behaviour of the system is governed by the poles of the system, that is, by the eigenvalues of matrix $\mathbf{A}$. It is often desirable to relocate the poles of the system in order to obtain particular dynamic behaviour such as stability. This can be achieved by using a state-feedback control

$$\mathbf{u}(t) = \mathbf{F}\mathbf{x}(t) \tag{2.2}$$

where $\mathbf{F} \in \Re^{m \times n}$, the feedback control gain matrix, is chosen such that the modified dynamic system

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{F})\mathbf{x}(t) \tag{2.3}$$

has the desired poles. Wonham [W3] in 1967 presented the fundamentals of the eigenvalue assignment using state feedback control. He showed that the poles of the system can be assigned by state feedback if the system is controllable. In other words, the solution for the feedback control gain matrix $\mathbf{F}$ exists if the pair $(\mathbf{A}, \mathbf{B})$ is completely controllable, that is:

$$\left\{\mathbf{v}^T\mathbf{A} = \sigma\mathbf{v}^T \text{and } \mathbf{v}^T\mathbf{B} = \mathbf{0}\right\} \Leftrightarrow \mathbf{v}^T = \mathbf{0} \tag{2.4}$$

If $(\mathbf{A}, \mathbf{B})$ is not controllable then there exists $\mathbf{v}^T \neq \mathbf{0}$ such that $\mathbf{v}^T\mathbf{A} = \sigma\mathbf{v}^T$ and $\mathbf{v}^T\mathbf{B} = \mathbf{0}$, then $\mathbf{v}^T(\mathbf{A} + \mathbf{B}\mathbf{F}) = \sigma\mathbf{v}^T$ for all the control gain $\mathbf{F}$. Thus the open-loop eigenvalues σ are the eigenvalues of the closed-loop system $(\mathbf{A} + \mathbf{B}\mathbf{F})$ for all $\mathbf{F}$ and it cannot be assigned to prescribed values by any feedback control.

If there exists a continuous input $\mathbf{u}(t)$ that transfers the initial state $\mathbf{x}(t_0)$ from zero to any final state $\mathbf{x}(t_f)$ within a finite time interval $t_f - t_0$ the system is said to be controllable [P5], [M6].

Controllability can also be examined by the rank deficiency of the controllability matrix that is;

$$\operatorname{rank}(\mathscr{C}) = n \tag{2.5}$$

where

$$\mathscr{C} = \left(\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\right). \tag{2.6}$$

In feedback control, the control forces are dependent on the physical measurements of the outputs $\mathbf{y} \in \mathfrak{R}^{l \times 1}$. The outputs are related to the state variables by the sensor distribution matrix $\mathbf{D} \in \mathfrak{R}^{l \times n}$,

$$\mathbf{y}(t) = \mathbf{Dx}(t) \tag{2.7}$$

In a similar manner, the system is observable if the observability matrix

$$\mathscr{O} = \begin{pmatrix} \mathbf{D} \\ \mathbf{DA} \\ \ldots \\ \mathbf{DA}^{n\text{-}1} \end{pmatrix} \tag{2.8}$$

is such that

$$\operatorname{rank}(\mathscr{O}) = n\,. \tag{2.9}$$

In this case, the system described by the pair $(\mathbf{A}, \mathbf{D})$ is said to be observable.

Numerous algorithms involving both state and output feedback control have been developed for the eigenvalue assignment problem. Davison [D10] generalised the earlier result of Wonham [W3] for the output feedback control. He showed that if the system is both controllable and observable, m poles of closed-loop system are assignable by gain output feedback, where m is the number of independent outputs. This result was extended by Kimura [K5] that if the system is controllable and observable and if $n \leq r + m - 1$, an almost arbitrary set of distinct closed-loop poles is assignable by gain output feedback, where, n , r and m are the numbers of state variables, inputs and outputs respectively. Miminis [M7] proposed an algorithm based on the deflation technique for eigenstructure assignment using state feedback. Saad [S1] used projection method for eigenvalue assignment of single-input control systems. In this approach the orthonormal left subspaces associated with the eigenvalues to be assigned are computed to determine the feedback gain.

In structural dynamics, it is preferable to work with the dynamic equations in the second-order form rather than in the first order state-space form, the reason being that the natural properties of the system matrices such as bandedness, definiteness and symmetry are lost after transforming into state-space form.

The second order dynamic equation may be written as,

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t) \tag{2.10}$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \Re^{n\times n}$ are symmetric and $\mathbf{v}^T\mathbf{M}\mathbf{v} > 0$, $\mathbf{v}^T\mathbf{C}\mathbf{v} \geq 0$, $\mathbf{v}^T\mathbf{K}\mathbf{v} \geq 0$ for arbitrary $\mathbf{v} \in \Re^{n\times 1}$. The feedback control force $\mathbf{u}(t)$ is linearly dependent on the displacements $\mathbf{x}(t)$ and velocities $\dot{\mathbf{x}}(t)$.

$$\mathbf{u}(t) = -\mathbf{F}^T\dot{\mathbf{x}}(t) - \mathbf{G}^T\mathbf{x}(t) \tag{2.11}$$

where $\mathbf{F}, \mathbf{G} \in \Re^{n \times m}$.

The eigenvalues of the open-loop system may be achieved by defining the roots of $\det(P(\sigma)) = 0$ where,

$$P(\sigma) = \sigma^2\mathbf{M} + \sigma\mathbf{C} + \mathbf{K} \tag{2.12}$$

is the quadratic pencil for the second-order system (2.10). The inverse problem of eigenvalue assignment requires us to find the feedback control gains $\mathbf{F}$ and $\mathbf{G}$ which assigns the closed-loop eigenvalues to prescribed values. In the eigenvalue assignment problem, there might be certain poles of a vibrating system which need to be reassigned such as those associated with instability. The other poles of the system can remain unchanged. The problem of reassigning some of the eigenvalues while keeping the remaining eigenvalues unchanged is called the *partial eigenvalue assignment*. Different iterative and non-iterative algorithms were suggested for the quadratic pencil. The technique suggested by Datta *et al.* [D6] is based on eigenvalue orthogonality conditions for the matrices **M**, **C** and **K**. The proof of orthogonality relationships are provided in [D3]. An algorithm was developed to derive the control gains for the case of single-input control [D7]. It was observed that a unique solution of the control gains exists for the single-input control. Similar algorithm of multi-input pole assignment problem resulted in a family solution for feedback gain matrices [D4]. The necessary and sufficient conditions for the existence and uniqueness of the solution for partial eigenvalue assignment were discussed. Ram *et al.* [R5] considered a multi-input partial pole assignment as a sequence of single-input control actions. The problem was solved by moving the poles gradually from their initial locations to their final destination. Ram and Elhay [R3] constructed a tri-diagonal, symmetric quadratic pencil for the inverse eigenvalue problem. The number of solutions for the cases of distinct and repeated eigenvalues was discussed. A new method was presented [D8] for the multi-input pole assignment in which the mass, stiffness and damping matrices were updated in order to assign the prescribed eigenvalues. No explicit knowledge of eigenvalues and eigenvectors of the open-loop quadratic pencil were required.

In addition to the eigenvalue assignment, there exists an extra freedom for assignment of eigenvectors. For example the transient response of the structure depends not only on the eigenvalues which determines the decay rate of the response but also on the eigenvectors which show the mode shapes. The right eigenvectors fix the shape of the mode and the left eigenvectors determine the amount each mode is excited in the response. Datta *et al.* [D5] considered eigenstructure assignment for the quadratic pencil by multi-input state feedback control. A complete parametric approach was presented in [D13] for the eigenstructure assignment using the proportional plus derivative feedback controller. Under the controllability conditions a complete parametric expressions for the closed-loop eigenvector matrix and the feedback gains were established in terms of the closed-loop eigenvalues and a group of parameter vectors. The main computations involved were two singular-value decompositions and manipulating the open-loop coefficient matrices. Schulz and Inman [S2] developed an eigenstructure assignment technique for vibration suppression and active vibration isolation. The analytical model of the system was used to minimize the response of the system. The objective function included vibration suppression and simplifying the controller design. The advantage of this technique is that minimizing the system response prevents the unassigned eigenvalues from becoming unstable; however the technique is restricted to small systems because of the large computations involved.

An eigenvalue embedding problem was proposed [C2] which preserved the symmetry of the system and was related to the finite element model updating problem. The eigenvalue embedding problem was concerned with updating the FE model in a way that the required eigenvalues or eigenvectors are assigned. Datta [D9] incorporated the measured eigenvalues and eigenvectors into the model using feedback control. The practical difficulty with this method is that a model reduction technique has to be applied to the FEM or the measured mode shapes have to be expanded [D8].

In eigenvalue assignment the stability of the closed-loop system is the major concern. To ensure stability, all the poles of the system must lie on the left-hand side of the complex plane. It is not always easy to determine whether all eigenvalues have negative real parts, especially when the system has a large number of degrees of freedom. In partial eigenvalue assignment while assigning some eigenvalues, it may happen that remaining eigenvalues, which are not assigned, are shifted towards the right-hand side of the complex plane, and therefore the system may lose its stability due to spillover. Balas [B1] investigated the effects of uncontrolled modes (residual modes), which could lead to spillover and instability in the closed-loop system. Robust eigenstructure assignment techniques were proposed to reduce spillover of the poles and improve the condition of the closed-loop system matrix.

2.2.2 Robust control

The inverse problem of pole assignment for the multi-input state feedback control may have a family of solutions for the control gains. Therefore, defining solutions in which the assigned poles are as insensitive to perturbations as possible is the aim of robustness analysis. It is known that the sensitivities of the eigenvalues of a matrix are dependent on the corresponding eigenvectors [W2]. Therefore, generation of well-conditioned eigenvectors is a key issue for the family of pole placement algorithms which results in an insensitive closed-loop eigenvalues with respect to perturbation of system parameters or control gains. Among the many published papers related to robustness theory, four methods address the problem of robustness and eigenvalue sensitivity to uncertainties. The first method is to determine the derivatives of system matrices with respect to uncertain parameters. The second approach is to optimize the index in an eigenvalue placement constraint in the form of Sylvester's equation. The third method involves the minimization of condition numbers and various other robustness measures. The fourth method employs orthogonal projections into linear subspaces of eigenvectors to improve the conditioning. Kautsky *et al.* [K3] introduced four iterative algorithms for robust solutions to the multi-input state-feedback pole assignment problem. All the algorithms use the concept of orthogonal projection into linear subspaces of eigenvectors to improve iteratively various equivalent measures of conditioning for robustness of the closed-loop system. In method '0' a rank-one update is made to eigenvector matrix $\mathbf{X}$ in a way that each updated vector $\mathbf{x}_j$ is as orthogonal as possible to the space spanned by the remaining vectors; therefore minimizing the condition $c_j = 1/\left|\tilde{\mathbf{y}}_j^T \mathbf{x}_j\right|$ where $\tilde{\mathbf{y}}_j^T$ is the normalized left eigenvector. Method '1' is similar to method 0; the difference being in the condition number in which method 1 uses a weighted sum of the squares of all the condition numbers. In methods '2' and '3', an orthonormal set of $\tilde{\mathbf{x}}_j, j = 1,2,\ldots,n$ is chosen such that some measure of distance between the vectors $\tilde{\mathbf{x}}_j$ and a specified subspace is minimized, and then the required eigenvectors $\mathbf{x}_j$ are obtained from the normalized projection of $\tilde{\mathbf{x}}_j$ into that subspace. However, the convergence of the above methods may not be guaranteed. Juang *et al.* [J1] presented a non-iterative algorithm based on Kautsky's technique for output feedback control system using the design freedom to choose the minimum control gain when the number of assigned eigenvalues is less than the number of assignable eigenvalues. Rew *et al.* [R7] employed projections onto subspaces of admissible eigenvectors which utilizes unitary eigenvectors as the desired or targeted eigenvectors and the optimal eigenvectors were determined in a least-square sense. Kautsky and Nichols [K4] presented an efficient and reliable numerical method for minimizing the pole sensitivity to structured perturbations. It was demonstrated that after a small number of iterations the improvement of the sensitivity measure was achieved when the system was subjected to random perturbations. Juang *et al.* [J2] considered robust eigenstructure assignment using second order models, a second order adaptation of the well-known robust eigenvalue assignment method by Kautsky *et al.* [K3] for first-order systems. Robustness was achieved by choosing the eigenvectors as close as possible to the column space of a well-conditioned matrix. Chu [C6] modified the method proposed by Juang and Maghami [J2]. In his approach robustness was achieved by minimizing some condition numbers of the eigenvalues of the closed-loop second order pencil. A general eigenstructure assignment for a second order linear system via proportional-derivative plus partial second-order state feedback was presented [D14]. The sensitivity measures included minimization of the condition number, feedback gains and the sensitivity of the closed-loop eigenvalues.

A new approach to robust stabilization of second order systems with proportional-derivative feedback was introduced by Henrion *et al.* [H1]. The robust pole placement was performed based on developed sufficient conditions for stability of polynomial matrices and a linear matrix inequality (LMI).

2.2.3 *Active damping strategies*

Various methods have been applied to vibration control in the engineering field. Traditionally, passive isolators and dampers are used to suppress the mechanical vibration. Recent advances in digital signal processing and sensor and actuator technology have attracted the vibration and control community. Active damping control has been studied to avoid structures from resonating with high amplitude at their natural frequencies. Among the many proposed techniques in active vibration control the most recent ones are chosen here for the literature survey. Preumont addressed the process of designing an active control system in detail [P5]. It was shown that the use of collocated actuators and sensors (e.g. physically located at the same place and energetically conjugated, such as force and displacement/velocity or torque and angle) leads to an alternating pole/zero pattern. If the structure is undamped, the line of interlacing poles and zeros is on the imaginary axis and if the structure is lightly damped, it is slightly on the left half plane. This property of interlacing poles and zeros guarantees the stability of the control systems because the root locus plot remains entirely within the left half-plane. The collocated actuators and sensors can also be implemented in a decentralized manner that every actuator interacts only with its collocated sensor. For non-collocated actuators and sensors this interlacing property however, no longer holds and the root locus plot may exhibit so-called 'pole-zero flipping' when the system parameters are changed slightly. Cannon and Rosenthal [C1] showed by experiments that in the case of non-collocated sensors and actuators, any controlled flexible system may be extremely sensitive to system parameters and may require sophisticated techniques to achieve robust control.

Active damping by velocity feedback was considered by Gardonio *et al.* in a three-part paper [G1], [G2], [B8]. The theory, design and application of a smart panel, which comprised 16 decentralised units for the control of sound transmission was addressed. Each control unit consisted of a collocated accelerometer-sensor and a piezoceramic patch actuator with a single channel velocity feedback controller to generate active damping. Gardonio and Elliott [G3] carried out a theoretical study on the active structural acoustic control of a new smart panel with sixteen triangularly shaped piezoelectric patch actuators. The actuators were distributed along the perimeter of the panel, and velocity sensors were positioned at the vertices opposite the base edges. The performance was assessed and contrasted with that of a conventional smart panel using an array of square piezoelectric patch actuators. The two control systems generated active damping to reduce the response and sound radiation of the panel in the lightly damped and well separated low-frequency resonances. The same authors [G4] analysed the theory of flexural vibration of a beam with a direct velocity feedback using either an ideal collocated force actuator or a closely located piezoelectric patch actuator. The effect of increasing the control gain on the vibration of the beam was analysed.

Preumont *et al.* [P4] considered active damping by a local force feedback using piezoelectric actuators and showed a significant increase of damping in the first mode with one actuator. Active vibration damping using inertial actuators with local displacement feedback control was considered by Benassi and Elliott [B4]. The inertial actuator was designed to have a low resonance frequency, thus preventing the unwanted static deflection of the actuator and the local displacement feedback loop was used to provide self-levelling for the actuator. Similar approach for active vibration damping using inertial actuator was taken with direct and integrated force displacement feedback [B6]. It was shown that the direct force feedback loop lowered the natural frequency of the actuator however, made the stability of the loop more sensitive to phase shifts. The integrated force feedback loop increased the stability by rotating the Nyquist plot by $+90^{\circ}$, which made the control system more robust. A phase-lag compensator was designed, which combined both advantages of direct and integrated force feedback.

A different approach, active constrained layer damping [S7], allows us not only to reduce the spillover in the high frequency vibration modes by passive elements but also to control the vibration amplitude of low frequency modes by active elements. In active constraint layer damping, an actuator, usually in the form of a piezoelectric layer, is added to a conventional passive constrained layer damper, thus combining the best features of passive and active control in structural vibration.

2.2.4 Independent modal space control

Considerable attention has been directed recently towards the design of active vibration control systems for large flexible structures. An exact description of the dynamics of a flexible system requires an infinite number of degrees of freedom. For practical implementations, only a finite number of terms are modelled by truncation of modes to approximate the dynamics of the system. Control systems based on discretized models must be able to cope with truncation effects and spillover, which are the inevitable consequences of the uncontrolled and unmodelled modes. The techniques employed in the design of such control systems are primarily based on the *modal control* methods [P2] whereby the dominant modes of the flexible structure are controlled.

The dynamic equation of the linear time-invariant system takes the form of a set of n second-order differential equations.

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{f}(t) \tag{2.13}$$

The mass matrix $\mathbf{M}$ is assumed to be non-singular. A transformation of $\mathbf{x} = \mathbf{M}^{-1/2}\mathbf{q}$ is considered to mass-normalize the damping and stiffness matrices [I1]. Pre-multiplying equation (2.13) by $\mathbf{M}^{-1/2}$ yields;

$$\ddot{\mathbf{q}} + \widetilde{\mathbf{C}}\dot{\mathbf{q}} + \widetilde{\mathbf{K}} = \mathbf{M}^{-1/2}\mathbf{f} \tag{2.14}$$

Here the coefficients are

$$\widetilde{\mathbf{C}} = \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2} \quad \text{and} \quad \widetilde{\mathbf{K}} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2} \qquad (2.15),(2.16)$$

The matrices as constructed in (2.15) and (2.16) are symmetric with equal right and left eigenvectors. The transformation matrix $\mathbf{P}$ is formed by combining the normalized eigenvectors as columns. Substituting $\mathbf{q} = \mathbf{P}\mathbf{z}$ into (2.14) and pre-multiplying by the transpose of $\mathbf{P}$, the n dynamic equations in modal coordinates are derived as;

$$\ddot{\mathbf{z}}_i + \mathbf{P}^T\widetilde{\mathbf{C}}\mathbf{P}\dot{\mathbf{z}}_i + \mathbf{P}^T\widetilde{\mathbf{K}}\mathbf{P}\mathbf{z}_i = \mathbf{P}^T\mathbf{M}^{-1/2}\mathbf{f} \tag{2.17}$$

The necessary and sufficient condition for the decoupling of the equation (2.17) is found to be [C6];

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \tag{2.18}$$

Thus with condition (2.18) satisfied, the system can be analysed in modal space.

$$\ddot{\mathbf{z}}_i + 2\zeta_i\omega_i\dot{\mathbf{z}}_i + \omega_i^2\mathbf{z}_i = \mathbf{P}^T\mathbf{M}^{-1/2}\mathbf{f} \tag{2.19}$$

where

$$\begin{aligned}\mathbf{P}^T\widetilde{\mathbf{C}}\mathbf{P} &= 2\mathrm{diag}(\zeta_i\omega_i)\\ \mathbf{P}^T\widetilde{\mathbf{K}}\mathbf{P} &= \mathrm{diag}(\omega_i^2)\end{aligned} \qquad (2.20),(2.21)$$

Here, ζ_i and ω_i are the modal damping ratios and frequencies respectively and the subscript refers to the i th element of the vector $\mathbf{z}$.

The special case of the proportional damping matrix characterised by the expression,

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{2.22}$$

may be shown to satisfy the condition (2.18). In real structures, however, one may find different forms of damping, which generally will not be proportional to mass and stiffness.

The idea behind modal control is to use the transformation $\mathbf{P}$ to decouple the equations and then choose the individual control inputs to control one mode without affecting the others. However, the closed-loop equations of the system are coupled via the feedback control and defining the feedback control gains requires the solution of a coupled matrix Ricatti equation. For large flexible structures the resulting Ricatti equation can pose serious computational difficulties. This modal transformation provides for so-called internal decoupling. However, in feedback control, modal transformation does not necessarily yield completely decoupled equations, since there may still be external coupling among the modes via the feedback control. Meirovitch and his students [M6] developed a method referred to as '*independent modal space control*', which is based on the classical modal control, in which complete decoupling of the controlled modes is achieved. The control laws designed in the modal space depend only on the corresponding modal coordinates and therefore do not recouple other coordinates.

If we rearrange the second order dynamic equation into the state-space form;

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2.23}$$

$$\mathbf{y}(t) = \mathbf{D}\mathbf{x}(t) \tag{2.24}$$

The matrix $\mathbf{A}$ is not diagonal but block diagonal, with the order of blocks being 2. If the modal feedback forces $\mathbf{u}_i(t)$ depend only on $\mathbf{x}_i(t)$, as $\mathbf{u}_i = \mathbf{u}_i(\mathbf{x}_i)$, then external decoupling may also be achieved. The equation (2.19) can be partitioned into controlled and uncontrolled (residual) modes.

$$\dot{\mathbf{x}}_C = \mathbf{A}_C\mathbf{x}_C + \mathbf{B}_C\mathbf{u}_C \tag{2.25}$$

$$\dot{\mathbf{x}}_R = \mathbf{A}_R\mathbf{x}_R + \mathbf{B}_R\mathbf{u}_R \tag{2.26}$$

where the subscripts C and R in equations (2.25) and (2.26) correspond to the controlled and residual modes respectively. In a proportional control, the relationship between the modal control and the state vector can be defined as;

$$\mathbf{u} = -\mathbf{G}\mathbf{x} \tag{2.27}$$

where $\mathbf{G}$ is the modal control gain. Partitioning the matrix $\mathbf{G}$;

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \tag{2.28}$$

and rewriting the equations (2.25) and (2.26) yields;

$$\mathbf{u}_C = -\left(\mathbf{G}_{11}\mathbf{x}_C + \mathbf{G}_{12}\mathbf{x}_R\right) \tag{2.29}$$

$$\mathbf{u}_R = -\left(\mathbf{G}_{21}\mathbf{x}_C + \mathbf{G}_{22}\mathbf{x}_R\right) \tag{2.30}$$

If the residual modes are not to affect the controlled modes then the sub-matrix of the control gain $\mathbf{G}_{12}$ should be zero. It was shown [M6] that other strict requirements are enforced for the other sub-matrices of control gain $\mathbf{G}$ to achieve the decoupling of the closed-loop system. It can be concluded that in the independent modal space control the flexibility in choosing the control gains is restricted by the decoupling requirement.

Another drawback with this approach is the transformations from the physical space to modal space and vice versa. In practice the control forces are applied in physical coordinates and are obtained from the inverse transformation of modal coordinates to physical coordinates. The physical control forces must ensure the control of one mode independently of the other modes, which necessitates the use of a sufficient number of actuators. Modern distributed actuators can be designed to control the vibration modes of simple structures such as plates; however this technique is not applicable to large-scale built-up structures consisting of many components. The states of the system are also measured in physical coordinates and are transformed into modal coordinates using observers. A Luenberger observer or a Kalman filter can be used to estimate the full controlled modal state $\mathbf{x}_C(t)$ from the output $\mathbf{y}(t)$.The dynamic equation of the modal observer is given in [M6], [P5],

$$\dot{\hat{\mathbf{x}}}_C = \mathbf{A}_C\hat{\mathbf{x}}_C + \mathbf{B}_C\mathbf{u} + \mathbf{K}(\mathbf{y} - \mathbf{C}_C\hat{\mathbf{x}}_C) \tag{2.31}$$

The interaction between the controlled system and the residual modes was addressed by Balas [B1] and was analysed by considering the equation formed by the state variables $\left(\mathbf{x}_C^T, \mathbf{x}_R^T, \mathbf{e}_C^T\right)^T$,

$$\begin{pmatrix} \dot{\mathbf{x}}_C \\ \dot{\mathbf{x}}_R \\ \dot{\mathbf{e}}_C \end{pmatrix} = \begin{bmatrix} \mathbf{A}_C - \mathbf{B}_C\mathbf{G}_C & \mathbf{0} & -\mathbf{B}_C\mathbf{G}_C \\ -\mathbf{B}_R\mathbf{G}_C & \mathbf{A}_R & -\mathbf{B}_R\mathbf{G}_C \\ \mathbf{0} & \mathbf{K}\mathbf{C}_R & \mathbf{A}_C - \mathbf{K}\mathbf{C}_C \end{bmatrix} \begin{pmatrix} \mathbf{x}_C \\ \mathbf{x}_R \\ \mathbf{e}_C \end{pmatrix} \tag{2.32}$$

where $\mathbf{e}_C = \mathbf{x}_C - \hat{\mathbf{x}}_C$. In this equation, the control spillover arises via the term $-\mathbf{B}_R\mathbf{G}_C$, which is responsible for the excitation of the residual modes by the control force. The observation spillover arises from the sensor output being contaminated by the residual modes via the term $\mathbf{KC}_R$. If either one of the terms $\mathbf{C}_R$ and $\mathbf{B}_R$ become zero, the eigenvalues of the closed-loop system are determined from the diagonal terms of the matrix in equation (2.32). It was shown that the control spillover does not destabilize the system although it can cause degradation in the system performance. However observation spillover can cause instability in the closed-loop eigenvalues [M4]. Therefore, the design of an accurate observer is essential to reduce the observation spillover.

A modified independent modal space control (IMSC) is developed to account for the control spillover due to the use of fewer actuators than modelled modes [B2]. In this method modal forces are taken into account that would otherwise excite the residual modes generated by spillover. A time-sharing approach using a small number of actuators to control a large number of modes can be effective in suppressing vibration if the actuators are dedicated to control the modes that have the highest modal energy at any particular instant. It was shown that the modified independent control is more effective than IMSC by series of experimental results [B3].

Stobener and Gaul [S8] applied the method of independent modal space control to structures with more complex geometries including a car body. The system eigenvalues and eigenvectors were obtained from experimental modal analysis.

2.3 Conclusion

In this chapter an overview of structural modification and active vibration control literature is provided. A survey of the structural modification literature is included, covering forward modification, inverse problems and measurement of the rotational receptances. In this section the main focus is on inverse problems of eigenvalue assignment.

Active vibration control strategies such as eigenvalue assignment, robust control, active damping and independent modal space control have received great attention from the structural dynamics and active vibration control communities. A full description of these techniques is included.

Chapter 3

STRUCTURAL MODIFICATION THEORY

3.0 Introduction

In this chapter the general theory of structural modification based on the paper by Mottershead and Ram [M14] is presented. The principles of structural modification for passive modification and pole placement by active vibration control are the same. The effect of structural modifications on the system is defined by using the receptances of the original system at the modification coordinates; this is known as the receptance method. The rank 1 modification theory includes the Vincent Circle method. General modifications of higher rank order are also described. Numerical examples are included to demonstrate how inverse problems of assigning poles and zeros in structural modification may be solved.

3.1 Introductory theory

The theory of structural modification presented in this chapter can be applied to both passive modification and active vibration control. Passive modification is traced back to the work of Duncan [D15], who in 1941 determined the dynamic behaviour of a compound system formed from two or more subsystems with known receptances and interconnection properties. A typical inverse problem might be to assign a number of poles and zeros by modifying the structure, such as by adding point masses, springs, beams or plates. In this way the natural frequencies of a structure may be shifted to desired locations, or antiresonances moved so that the vibration response vanishes at chosen coordinates and frequencies.

State feedback control may be traced back to Wonham [W3] who showed that if the system is controllable then all the poles may be assigned to arbitrarily prescribed locations. Kalman [K1] showed that a linear dynamical system could be an irreducible realization of an impulse-response matrix, if and only if the system was completely controllable and completely observable. Gilbert [G6] determined the controllability and observability of multivariable control systems.

The methodology presented here for the inverse problem of eigenvalue assignment is the same for passive modification and active vibration control. It is based on the receptance matrix $\mathbf{H}(s)$ which is by definition, the inverse of the dynamic stiffness matrix.

The theory of structural modification may be explained by starting with the homogenous equation of motion for the multi degree of freedom system,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{3.1}$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \Re^{n \times n}$ are symmetric and $\mathbf{v}^T\mathbf{M}\mathbf{v} > 0$, $\mathbf{v}^T\mathbf{C}\mathbf{v} \geq 0$, $\mathbf{v}^T\mathbf{K}\mathbf{v} \geq 0$ for arbitrary $\mathbf{v} \in \Re^{n \times 1}$.The displacement vector $\mathbf{x}(t)$ and its derivatives $\dot{\mathbf{x}}(t)$ and $\ddot{\mathbf{x}}(t)$ are time dependent real vectors. Taking Laplace transform of the second order differential equation yields the quadratic matrix pencil,

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{v} = \mathbf{0} \tag{3.2}$$

The $2n$ eigenvalues σ_i, $i = 1,2,\ldots,2n$ of the system are the roots of the characteristic polynomial equation,

$$\det\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right) = 0 \tag{3.3}$$

The eigenvector $\mathbf{v}_i$ is associated with the eigenvalue σ_i. The quadratic pencil can be written in terms of its $2n$ eigenpairs,

$$\mathbf{MV\Sigma}^2 + \mathbf{CV\Sigma} + \mathbf{KV} = \mathbf{0} \tag{3.4}$$

where,

$$\mathbf{\Sigma} = \mathrm{diag}\{\sigma_1 \quad \sigma_2 \quad \ldots \quad \sigma_{2n}\} \in \mathbb{C}^{2n\times 2n} \text{ and } \mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_{2n}] \in \mathbb{C}^{n\times 2n}$$

(3.5), (3.6)

The quadratic pencil may be transformed to the first-order standard form.

$$\mathbf{A\Phi} = \mathbf{\Phi\Sigma} \tag{3.7}$$

where,

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \text{ and } \mathbf{\Phi} = \begin{bmatrix} \mathbf{V} \\ \mathbf{V\Sigma} \end{bmatrix} \qquad (3.8),(3.9)$$

and $\mathbf{I}$ is the identity matrix of dimension n. For this system, the solution of equation is in the form,

$$\mathbf{x}(t) = \sum_{i=1}^{2n} a_i \mathbf{v}_i e^{\sigma_i t} \tag{3.10}$$

where a_i, $i = 1,2,\ldots,2n$ are the arbitrary constants which can be obtained from the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0$. The motion of the system can be determined from the eigenvalues. The imaginary part of the eigenvalue determines the frequency of oscillation. If the real part of each eigenvalue is negative the motion of the system decays in time indicating a stable system. When the real part of an eigenvalue vanishes then the system is marginally stable and oscillates everlastingly. In the case of a positive real part for any eigenvalue, the motion of the system increases and the system becomes unstable. The dynamics of the system is therefore defined by the position of the eigenvalues or poles of the system in the complex plane. One important objective is to shift the poles of the system to the left by either passive modification which is physically changing system properties or by active control which is applying external forces based on the real-time measurements of the system states.

The dynamic equations of motion for a forced vibratory system may take the form of,

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{f}(t) \tag{3.11}$$

where $\mathbf{f}(t)$ is the applied force to the q^{th} dof of the system. Taking the Laplace transform from Eq. (3.11) yields,

$$\mathbf{Z}(s)\mathbf{x}(s)=\mathbf{f}(s) \tag{3.12}$$

where s is the frequency in the complex plane having the units of time^{-1} and

$$\mathbf{Z}(s)=\left(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K}\right)\in\mathbb{C}^{n\times n} \tag{3.13}$$

is called the dynamic stiffness matrix. The inverse of dynamic stiffness matrix $\mathbf{Z}(s)$ is defined as receptance matrix and denoted by,

$$\mathbf{H}(s)=\left(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K}\right)^{-1} \tag{3.14}$$

In practice receptances $\mathbf{H}(s)$ may be obtained from the measured receptance frequency response function $\mathbf{H}(\mathrm{i}\omega)$.

The dynamic equations in the form of dynamic stiffness matrix Eq. (3.12) requires measurement of all the states $\mathbf{x}\in\mathbb{C}^{n\times 1}$. Multiplying Eq. (3.12) by the inverse of the dynamic stiffness matrix $\mathbf{Z}^{-1}(s)$, leads to the dynamic equations in the form of receptance matrix,

$$\mathbf{x}(s) = \mathbf{H}(s)\mathbf{f}(s) \tag{3.15}$$

In this equation, the receptance matrix is multiplied by the non-zero terms in $\mathbf{f}$, here is the q^{th} dof of the system, which allows us to accomplish structural modification with only a small number of states $\mathbf{x}$. Therefore the receptance method avoids model reduction or the use of observers applied to large FE models. In addition FE models may have inaccuracies in system parameters particularly in the damping matrix. The boundary conditions and the joints in FE models may not represent those in a physical structure. Therefore the measured receptance matrix is to be preferred over the receptance matrix obtained by the FE model. The objective of this research is to introduce the measured receptances in the structural modification theory.

The receptance $h_{pq}(s)$ describes the relation between the force applied to the q^{th} dof and the displacement response at p^{th} dof.

$$h_{pq}(s) = \mathbf{e}_p^T \mathbf{H} \mathbf{e}_q \tag{3.16}$$

where $\mathbf{e}_p$ and $\mathbf{e}_q$ are the unit vectors with unity at coordinates p and q respectively and zero terms elsewhere.

3.2 Dynamic absorption

Vibration absorption is the inverse problem of defining a simple modification which produces an antiresonance at a prescribed frequency. If $h_{pq}(s) = 0$ the dynamic absorption occurs at the complex frequency s.

According to Kramer's rule:

$$\text{if } \mathbf{A}\mathbf{x} = \mathbf{e}_q \text{ then } \mathbf{x}_p = (-1)^{p+q} \frac{\det(\mathbf{A}_{pq})}{\det(\mathbf{A})} \tag{3.17}$$

where $\mathbf{A}_{pq}$ is obtained by deleting the p^{th} row and q^{th} column of the matrix $\mathbf{A}$. When Kramer's rule is applied to the Eq. (3.12), and for simplicity the damping matrix $\mathbf{C} = \mathbf{0}$, the receptance h_{pq} is obtained as,

$$h_{pq} = (-1)^{p+q} \frac{\det\left(\mathbf{K}_{pq} - \omega^2 \mathbf{M}_{pq}\right)}{\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right)} \tag{3.18}$$

where $\mathbf{K}_{pq}$ and $\mathbf{M}_{pq}$ are obtained by deleting the p^{th} row and q^{th} column of $\mathbf{K}$ and $\mathbf{M}$ respectively. The roots of the numerator of $h_{pq}(\omega)$ are the frequencies at which the vibration absorption occurs.

$$\det\left(\mathbf{K}_{pq} - \omega^2 \mathbf{M}_{pq}\right) = 0 \qquad \omega \neq \lambda_k\,,\ k = 1,2,\ldots 2n \tag{3.19}$$

A numerical example is presented to illustrate the vibration absorption inverse problem.

Example 3.1.

A 4 dof system is considered as shown in Figure.3.1. The spring constant ξ is required to absorb the harmonic response at x_4.

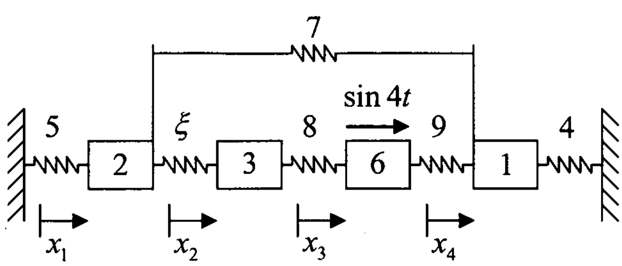

Figure 3.1. A 4 dof system with stiffness modification

The mass and stiffness matrices of the system are;

$$\mathbf{M}=\begin{bmatrix}2&0&0&0\\0&3&0&0\\0&0&6&0\\0&0&0&1\end{bmatrix},\qquad \mathbf{K}=\begin{bmatrix}12+\xi&-\xi&0&-7\\-\xi&8+\xi&-8&0\\0&-8&17&-9\\-7&0&-9&20\end{bmatrix} \tag{3.20}$$

The solution for the spring constant is obtained by solving,

$$\det\left(\mathbf{K}_{43}-\omega^2\mathbf{M}_{43}\right)=0 \tag{3.21}$$

Therefore,

$$\mathbf{K}_{43}-(4)^2\mathbf{M}_{43}=\begin{bmatrix}\xi-20&-\xi&-7\\-\xi&\xi-40&0\\0&-8&-9\end{bmatrix} \tag{3.22}$$

The real solution of the equations (3.21), (3.22) gives the spring constant of $\xi=14.8760$. This is the spring constant which assigns a zero at the receptance h_{43}.

3.3 Rank 1 modification

The unit-rank modification includes the addition and subtraction of concentrated masses, grounded springs and grounded dampers and springs and dampers connected between two coordinates.

The equation of motion of a dynamic system can be written in the usual form,

$$\mathbf{Z}(s)\mathbf{x}(s)=\mathbf{f}(s),\quad \mathbf{Z}(s)=\mathbf{K}+s\mathbf{C}+s^2\mathbf{M} \tag{3.23}$$

and a simple modification in the form of a point mass, grounded spring and grounded damper is connected at coordinate r, $z_r(s)=k_r+sc_r+s^2m_r$, where s is the complex frequency in radians/second. The modified dynamic stiffness matrix may be written in the form,

$$\mathbf{Z}(s)+\Delta\mathbf{Z}(s)=\mathbf{Z}(s)+z_r(s)\mathbf{e}_r\mathbf{e}_r^T \tag{3.24}$$

and matrix of receptances as,

$$\hat{\mathbf{H}}(s)=\left(\mathbf{K}+s\mathbf{C}+s^2\mathbf{M}+z_r(s)\mathbf{e}_r\mathbf{e}_r^T\right)^{-1} \tag{3.25}$$

where the circumflex denotes the modified system.

The inverse of a matrix with a unit-rank modification may be found in terms of the inverse of unmodified matrix by application of the Sherman-Morrison formula. The Sherman-Morrison formula, e.g. Golub and Van Loan [G7 p.51], describes the effect of a rank 1 modification on a matrix whose inverse is known.

$$\left(\mathbf{A}+\mathbf{b}\mathbf{c}^T\right)^{-1}=\mathbf{A}^{-1}-\frac{\mathbf{A}^{-1}\mathbf{b}\mathbf{c}^T\mathbf{A}^{-1}}{1+\mathbf{c}^T\mathbf{A}^{-1}\mathbf{b}} \tag{3.26}$$

The Sherman-Morrison formula has particular applications in determining modified system receptances as will now be explained.

The (unmeasured) modified-system receptances can be obtained in terms of the measured receptances of the original system and the known modification,

$$\hat{\mathbf{H}}(s)=\mathbf{H}(s)-\frac{z_r(s)\mathbf{H}(s)\mathbf{e}_r\mathbf{e}_r^T\mathbf{H}(s)}{1+z_r(s)\mathbf{e}_r^T\mathbf{H}(s)\mathbf{e}_r} \tag{3.27}$$

The pq^{th} term can then be selected from the matrix of modified receptances by,

$$\hat{h}_{pq}(s)=\mathbf{e}_p^T\hat{\mathbf{H}}(s)\mathbf{e}_q \tag{3.28}$$

which may be written explicitly in the form,

$$\hat{h}_{pq}(s)=\frac{h_{pq}(s)+z_r(s)\left(h_{pq}(s)h_{rr}(s)-h_{pr}(s)h_{rq}(s)\right)}{1+z_r(s)h_{rr}(s)} \tag{3.29}$$

Eq. (3.29) represents any term of the modified receptance matrix and can be simplified in the following cases:

1. A point receptance of the modified system, $p = q$,

$$\hat{h}_{qq}(s)=\frac{h_{qq}(s)+z_r(s)\left(h_{qq}(s)h_{rr}(s)-h_{qr}(s)h_{rq}(s)\right)}{1+z_r(s)h_{rr}(s)} \tag{3.30}$$

2. A cross receptance with modification at one of the coordinates, $p = r$,

$$\hat{h}_{rq}(s)=\frac{h_{rq}(s)}{1+z_r(s)h_{rr}(s)} \tag{3.31}$$

3. A point receptance with modification at the same coordinate, $p = q = r$,

$$\hat{h}_{rr}(s)=\frac{h_{rr}(s)}{1+z_r(s)h_{rr}(s)} \tag{3.32}$$

It can be seen that the poles of the system $\lambda_i, i=1,2,\cdots,2n$ are the roots of the characteristic equation,

$$1+z_r(\lambda_i)h_{rr}(\lambda_i)=0,\ i=1, 2,\ldots,2n \tag{3.33}$$

so that after re-arrangement the modification necessary to assign the i^{th} pole may be determined from,

$$\frac{1}{-z_r(\lambda_i)}=h_{rr}(\lambda_i) \tag{3.34}$$

Equation (3.34) may be used to assign an eigenvalue of the modified system by a passive modification of stiffness, damping and mass. For a conservative system in the case of a point mass modification Eq. (3.34) is simplified to become,

$$\frac{1}{\omega_i^2 m_r} = h_{rr}(\omega_i) \tag{3.35}$$

and for a grounded stiffness,

$$\frac{1}{-k_r} = h_{rr}(\omega_i) \tag{3.36}$$

where ω_i is the imaginary part of the pole λ_i. It is therefore possible to determine the unique solution for either k_r or m_r that assigns a particular value to a natural frequency. The zeros of the modified system may be obtained from the characteristic equation formed from the numerator of equation (3.29). A single cross-receptance zero μ_j may be assigned by using

$$\frac{1}{-z_r(\mu_j)} = h_{rr}(\mu_j) - \frac{h_{pr}(\mu_j)h_{rq}(\mu_j)}{h_{pq}(\mu_j)} \tag{3.37}$$

or a single point-receptance zero, from Eq. (3.30),by

$$\frac{1}{-z_r(\mu_j)} = h_{rr}(\mu_j) - \frac{h_{rq}^2(\mu_j)}{h_{qq}(\mu_j)} \tag{3.38}$$

where in both cases $r \neq p,q$.

If $p = q$ and the system is conservative then the zeros, like the poles, are located on the imaginary axis of the complex plane. In this case the zeros and poles interlace each other along the frequency axis of the point receptance $h_{qq}(\omega)$. Mathematically, the antiresonance frequencies are the eigenvalues of the adjoint system; the system obtained by deleting a row and a column from the original dynamic stiffness matrix. When $p \neq q$ then the dynamic stiffness matrix obtained by deleting the p^{th} row and q^{th} column may be asymmetric hence the eigenvalue μ_j will generally become complex.

Mottershead and Lallement [M9] showed how poles and zeros of an undamped (or lightly damped) structure could be shifted by a unit-rank modification. They established the necessary and sufficient conditions for the cancellation of a pole with a zero to produce a vibration node when the zeros were distinct.

Example3.2.

A 4 dof system is considered as shown in Figure.3.2. In the first problem, the additional stiffness between dofs 1 and 2 needed to assign a natural frequency at 4 rad/s is calculated and in the second problem, the additional stiffness required to assign a zero to the cross-receptance $h_{43}(s)$ at 4 rad/s is determined.

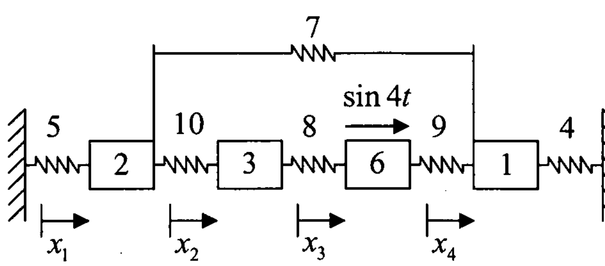

Figure 3.2. A 4 dof system

The case considered is that of a spring between two coordinates r and t. The matrix of modified-system receptances can be expressed as

$$\hat{\mathbf{H}}(s) = \mathbf{H}(s) - \frac{k_{rt}\mathbf{H}(s)(\mathbf{e}_r - \mathbf{e}_t)(\mathbf{e}_r - \mathbf{e}_t)^T \mathbf{H}(s)}{1 + k_{rt}(\mathbf{e}_r - \mathbf{e}_t)^T \mathbf{H}(s)(\mathbf{e}_r - \mathbf{e}_t)} \tag{3.39}$$

so that by application of Eq. (3.28) it is found that

$$\hat{h}_{pq}(s) = h_{pq}(s) - \frac{k_{rt}\left((h_{pr}(s) - h_{pt}(s))(h_{rq}(s) - h_{tq}(s)\right)}{1 + k_{rt}\left(h_{rr}(s) - h_{rt}(s) - h_{tr}(s) + h_{tt}(s)\right)} \tag{3.40}$$

In the present problem the coordinate indices are $r = 1$, $t = 2$. Then by setting the denominator to zero the spring that assigns a natural frequency of 4 rad/s is found to be,

$$-\frac{1}{k_{12}} = h_{11}(\lambda_i) - h_{12}(\lambda_i) - h_{21}(\lambda_i) + h_{22}(\lambda_i), \qquad \lambda_i = 4 \tag{3.41}$$

The additional spring stiffness, $k_{12} = 7.5648$, is found by entering the numerical values of the receptances at the chosen frequency. Figure 3.3(a) shows the initial and modified receptance $\hat{h}_{11}$ in decibels. The natural frequency of 4 rad/s is assigned after the added spring.

In the antiresonance assignment problem the receptance coordinates are $p = 4$ and $q = 3$. Eq. (3.40) is rearranged so that;

$$-\frac{1}{k} = h_{11}(\mu_i) - h_{12}(\mu_i) - h_{21}(\mu_i) + h_{22}(\mu_i) - \frac{h_{41}(\mu_i)h_{31}(\mu_i) - h_{41}(\mu_i)h_{23}(\mu_i) - h_{42}(\mu_i)h_{13}(\mu_i) + h_{42}(\mu_i)h_{23}(\mu_i)}{h_{43}(\mu_i)} \qquad \mu_j = 4 \tag{3.42}$$

After entering the numerical values of the receptances at $\mu_j = 4$, the value of $k = 4.8760$ is obtained confirming the result of the dynamic absorber in example 3.1. The modified receptance $\hat{h}_{43}(s)$ is shown in Figure 3.3(b) with an antiresonance at 4 rad/s as expected.

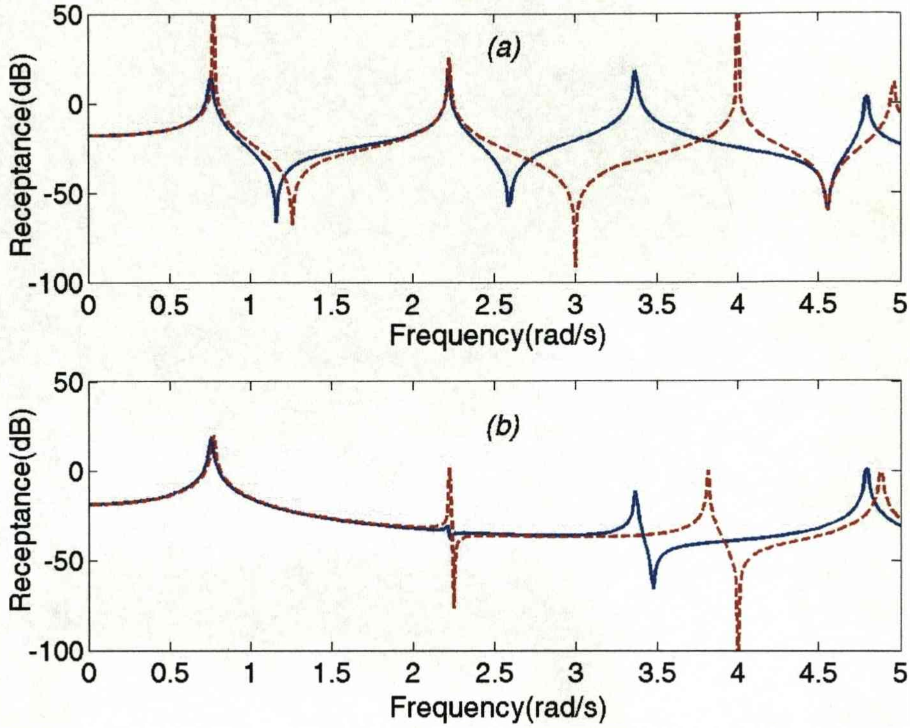

Figure 3.3. Original (full) and modified (dashed) receptances (a) assignment of the natural frequency at 4 rad/s to h_{11} (b) assignment of antiresonances at 4 rad/s to h_{43}

3.4 Vincent circle

Vincent's circle is used to define the frequency response of a structure at its arbitrary point. In 1972 Vincent [V2] discovered that when a structure is excited at a point q with a constant frequency and the system is modified between the two coordinates r and t, the response at point p traces out a circle when plotted in the complex plane as modification is varied from minus to plus infinity.

The system to which these equations apply is shown in Figure 3.4. It represents a system having many degrees of freedom for which it is required to find the response at point p due to force excitation at point q.

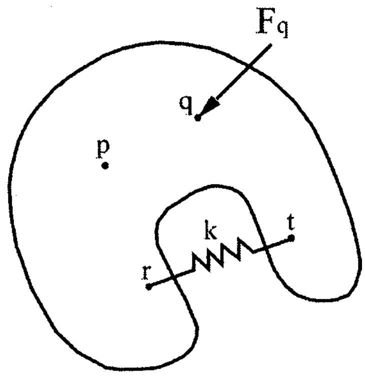

Figure 3.4. Structure with stiffness modification

A simple modification is applied by attaching a linear spring of stiffness k between two points r and t. The spring forces exerted at points r and t are $\mathbf{F}_r$ and $\mathbf{F}_t$ respectively, which have the relation,

$$\mathbf{F}_r = k(\mathbf{x}_t - \mathbf{x}_r) = -\mathbf{F}_t \tag{3.43}$$

The forcing vector $\mathbf{F}$ has three non zero elements $\mathbf{F}_r$, $\mathbf{F}_t$ and $\mathbf{F}_q$. The response of the original system before modification by the spring k at points p, r and t can be written as,

$$\mathbf{x}_p = h_{pq}\mathbf{F}_q + h_{pr}\mathbf{F}_r + h_{pt}\mathbf{F}_t \tag{3.44}$$

$$\mathbf{x}_r = h_{rq}\mathbf{F}_q + h_{rr}\mathbf{F}_r + h_{rt}\mathbf{F}_t \tag{3.45}$$

$$\mathbf{x}_t = h_{tq}\mathbf{F}_q + h_{tr}\mathbf{F}_r + h_{tt}\mathbf{F}_t \tag{3.46}$$

in which h_{ij} is the complex receptance at point i due to a force at point j. The forces $\mathbf{F}_r$ and $\mathbf{F}_t$ can be written in terms of $\mathbf{x}_r$ and $\mathbf{x}_t$ using Eqs. (3.44)-(3.46). Elimination of $\mathbf{x}_r$ and $\mathbf{x}_t$ and application of the Sherman-Morison formula [G7] leads to Eq (3.47) as illustrated in the previous section.

$$\frac{\mathbf{x}_p}{\mathbf{F}_q} = h_{pq} + \frac{k(h_{tq} - h_{rq})(h_{pr} - h_{pt})}{1 + k(h_{rr} - h_{tr} - h_{rt} + h_{tt})} \tag{3.47}$$

For the general case of modification $z_{rt} = k + cs$ connected between the two coordinates r and t, the modified receptance can be expressed as,

$$\hat{\mathbf{H}}(s) = \mathbf{H}(s) - \frac{z_{rt}(s)\mathbf{H}(s)(\mathbf{e}_r - \mathbf{e}_t)(\mathbf{e}_r - \mathbf{e}_t)^T \mathbf{H}(s)}{1 + z_{rt}(s)(\mathbf{e}_r - \mathbf{e}_t)^T \mathbf{H}(s)(\mathbf{e}_r - \mathbf{e}_t)} \tag{3.48}$$

The pq^{th} term can be obtained from the matrix of modified receptance by,

$$\hat{h}_{pq} = \mathbf{e}_p^T \hat{\mathbf{H}}(s)\mathbf{e}_q \tag{3.49}$$

Application of Eq. (3.49) to Eq. (3.48) yields;

$$\hat{h}_{pq}(s) = h_{pq}(s) - \frac{z_{rt}(s)((h_{pr}(s) - h_{pt}(s))(h_{rq}(s) - h_{tq}(s))}{1 + z_{rt}(s)(h_{rr}(s) - h_{rt}(s) - h_{tr}(s) + h_{tt}(s))} \tag{3.50}$$

Analysis of Vincent Circle is simplified by writing the equation (3.50) in the general form of,

$$a = b + \frac{zc}{1 + zd}, \quad a, b, c, d \in \mathbb{C} \tag{3.51}$$

where,

$$
\begin{aligned}
a &= \hat{h}_{pq} \\
b &= h_{pq} \\
c &= (h_{pr} - h_{pt})(h_{tq} - h_{rq}) \\
d &= (h_{rr} - h_{rt} - h_{tr} + h_{tt})
\end{aligned}
\tag{3.52}
$$

and $\mathbb{C}$ denotes the set of complex numbers. Equation (3.51) is non-linear in the modification parameter. Application of Vincent's circle provides a useful aid to analysis since the solution is restricted to the perimeter of a circle.

Rearranging equation (3.51) gives,

$$
z = \frac{a-b}{c+bd-ad} \tag{3.53}
$$

where $z \in \mathbb{C}$.

$a = b$, when $z = 0$

and

$a = b + \dfrac{c}{d}$, when $z = \infty$.

If the imaginary part of z is zero, a will be on the perimeter of a circle. The point closest to the origin of the complex plane defines the minimum value of a, which corresponds to the greatest suppression of vibration as shown in Figure 3.5.

$$
\min(a) = \xi\left(1 - \frac{\rho}{|\xi|}\right) \tag{3.54}
$$

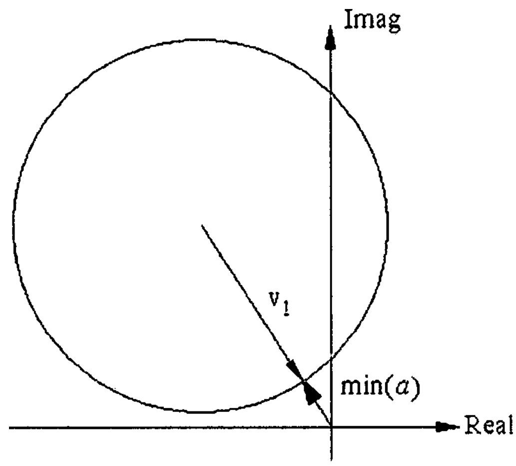

Figure 3.5. Vincent's Circle

where, ξ is the centre and ρ is the radius of the circle. The centre and radius of the circles varies with different modifications.

In the antiresonance assignment problem after rearranging the equation (3.51) the numerator is set to zero. The value of the modification obtained corresponds to the exact antiresonance assignment, but the modification is a complex number.

Application of Vincent's circle cannot assign an exact zero, however vibration will be minimised by a real valued modification obtained from the closed form solution presented by Mottershead and Ram [M14]. The theory of Vincent's circle was extended by Ghandchi Tehrani *et al.* [G5] to include mass, stiffness and damping modifications at different coordinates.

Example 3.3.

Consider the four-degree-of-freedom system as shown in Figure 3.2. The stiffness modification is applied between the coordinates 1 and 2. Viscous damping $\mathbf{C} = 0.01\mathbf{K}$ is considered. The modification, which suppresses the vibration at $h_{43}(\omega)$ is required.

The receptance matrix is determined using $\mathbf{H}(\omega) = (\mathbf{K} + \mathrm{i}\mathbf{C}\omega - \omega^2\mathbf{M})^{-1}$. The stiffness modification $k^* = 4.8768 - 0.5859\mathrm{i}$ is found to assign a zero at 4 rad/sec. The modification includes an imaginary part because proportional damping $\mathbf{C} = 0.01\mathbf{K}$ is applied to the original system. Using a real value of the modification doesn't assign an exact zero however, the vibration can be minimised by the closed form solution [M14]. The modification, which assigns a minimum vibration for receptance h_{43} is found to be 4.7610. Figure 3.6 shows the receptances h_{43} for the original and modified system.

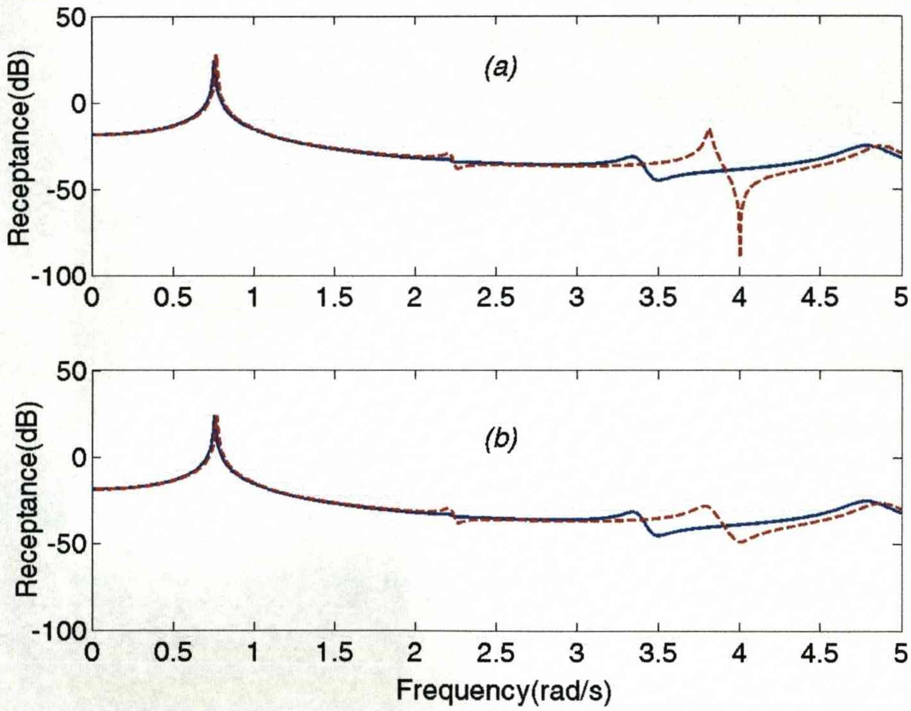

Figure 3.6. Original (full) and modified (dashed) receptances (a) exact assignment of the antiresonance (b) closest point on the circle

The circle is plotted as shown in Figure 3.7 using MATLAB. The centre of the circle is marked with '$*$', the origin of the complex plane with '$\times$', the point $k^* = 0$ with a small circle and the closest point to the origin which presents the minimum vibration with a dot. The positive range of k^* is plotted as a full line and the negative range as a dashed line. The value of k^* corresponding to the point on the circle closest to the origin of the complex plane is 4.7610.

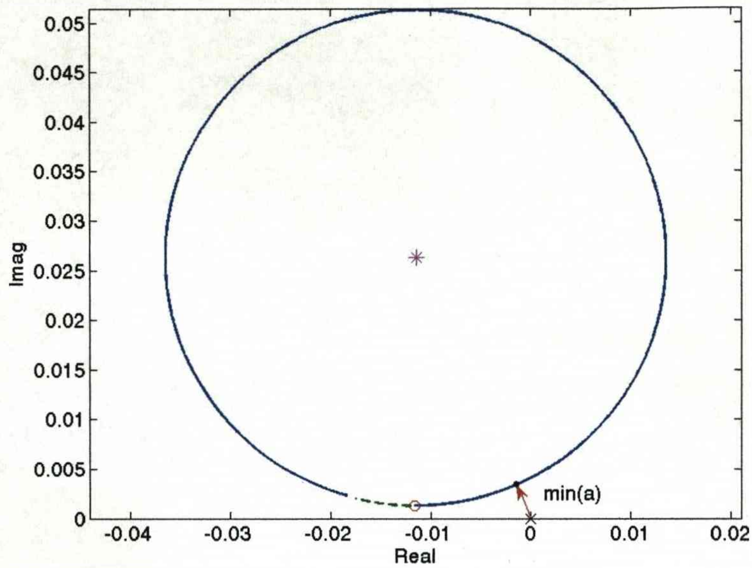

Figure 3.7. Vincent circle for receptance h_{43} with stiffness modification

3.5 General modification

Woodbury extended the Sherman-Morrison formula for the general case of a modification of arbitrary rank [W4]. The Woodbury formula may be stated as;

$$\left(\mathbf{A}+\mathbf{B}\mathbf{C}^T\right)^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{I}+\mathbf{C}^T\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}^T\mathbf{A}^{-1} \qquad (3.55)$$

In this formula, the matrix $\mathbf{BC}^T$ represents the modification. In structural dynamics, Sherman-Morrison-Woodbury formula is applied to determine the modified matrix in terms of the unmodified matrix and the general modification. The effect of structural modification on the system receptances can be determined from the original system receptances at the modification coordinates.

Another approach, used here, is to determine the modified receptance matrix from the general modification $\Delta\mathbf{Z}(s)$ in the dynamic equations. The assignment of multiple poles and zeros can be achieved by defining the terms in the modification matrix $\Delta\mathbf{Z}(s)$. In the general case of a modification of arbitrary rank,

$$\Delta\mathbf{Z}(s) = \Delta\mathbf{K} + s\Delta\mathbf{C} + s^2\Delta\mathbf{M} \tag{3.56}$$

so that the dynamic equation of the modified system may be expressed as,

$$(\mathbf{K} + s\mathbf{C} + s^2\mathbf{M})\mathbf{x}(s) = -\Delta\mathbf{Z}(s)\ \mathbf{x}(s) + \mathbf{f}(s) \tag{3.57}$$

The modification term $\Delta\mathbf{Z}(s)\mathbf{x}(s)$ in Eq. (3.57) can be treated as a forcing term on the unmodified structure. Multiplying both sides of the Eq. (3.57) by $\mathbf{H}(s) = (\mathbf{K} + s\mathbf{C} + s^2\mathbf{M})^{-1}$ and re-arranging yields;

$$(\mathbf{I} + \mathbf{H}(s)\Delta\mathbf{Z}(s))\mathbf{x}(s) = \mathbf{H}(s)\ \mathbf{f}(s) \tag{3.58}$$

The matrix of modified receptances can be defined in terms of the initial receptance and the general modification,

$$\hat{\mathbf{H}}(s) = \frac{\mathrm{adj}(\mathbf{I} + \mathbf{H}(s)\Delta\mathbf{Z}(s))\mathbf{H}(s)}{\det(\mathbf{I} + \mathbf{H}(s)\ \Delta\mathbf{Z}(s))} \tag{3.59}$$

The effect of the modification $\Delta\mathbf{Z}(s)$ using Eq. (3.59) is exactly the same as the effect of the modification $\mathbf{BC}^T$ using the Woodbury formula, Eq.(3.55), on the modified receptance matrix. Although the Woodbury formula for the case of $\mathbf{B} = \mathbf{C}$ gives a symmetric modified receptance matrix $\hat{\mathbf{H}}(s)$, Eq. (3.59) does not reveal this symmetry in the structure of the equation even for a symmetric modification $\Delta\mathbf{Z} = \mathbf{CC}^T$. However, both methods produce identical results.

The poles of the modified system, Eq. (3.59), are defined as the roots of the characteristic equation,

$$\det(\mathbf{I}+\mathbf{H}(\lambda_i)\,\Delta\mathbf{Z}(\lambda_i))=0, \quad i=1,\ldots,n \tag{3.60}$$

The eigenvectors corresponding to each of the eigenvalues are obtained from solving Eq. (3.61).

$$(\mathbf{I}+\mathbf{H}(\lambda_i)\Delta\mathbf{Z}(\lambda_i))\mathbf{x}_i(\lambda_i)=\mathbf{0}, \quad i=1,\ldots,n \tag{3.61}$$

The zeros are given from the terms of the matrix product in the numerator. For example the zeros of the pq^{th} modified receptance are given by the solution of,

$$[\mathrm{adj}(\mathbf{I}+\mathbf{H}(\mu_j)\Delta\mathbf{Z}(\mu_j))\mathbf{H}(\mu_j)]_{pq}=0 \tag{3.62}$$

where the subscript pq denotes the pq^{th} element of the matrix in square brackets.

Example 3.4.

In the 2 dof system shown in Figure 3.8 it is required to assign a natural frequency at 3 rad/sec and an antiresonance of $h_{22}(s)$ at 6 rad/sec using a spring k between the two masses and a mass m at coordinate 2.

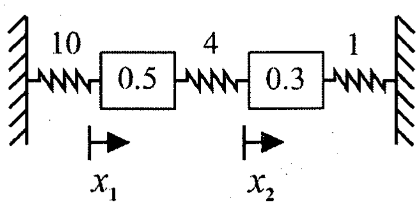

Figure 3.8. A 2 dof system

Proportional damping $\mathbf{C} = 0.0001\mathbf{K}$ is considered for simplicity. For the assignment of antiresonance at 6 rad/sec, Eq. (3.62) is solved.

$$\mathrm{adj}(\mathbf{I}+\mathbf{H}(s)\Delta\mathbf{Z}(s))\mathbf{H}(s) = \begin{bmatrix} h_{11} + (h_{11}h_{22} - h_{21}h_{12})(s^2 m + k) & h_{12} + k(h_{11}h_{22} - h_{21}h_{12}) \\ h_{21} + k(h_{11}h_{22} - h_{21}h_{12}) & h_{22} + k(h_{11}h_{22} - h_{21}h_{12}) \end{bmatrix} \tag{3.63}$$

For the assignment of natural frequency at 3 rad/sec, Eq. (3.60) is considered. Therefore,

$$\det(\mathbf{I}+\mathbf{H}(s)\,\Delta\mathbf{Z}(s)) = 1 + (h_{11} - h_{12} - h_{21} + h_{22})k + h_{22}s^2 m + (h_{11}h_{22} - h_{12}h_{21})s^2 mk \tag{3.64}$$

When the receptances of the original system are determined at the frequencies of 6 and 3 rad/s and entered into Eqs. (3.63) and (3.64), respectively, we obtain two complex equations in k and m.

$$(0.6171 + 0.0136\mathrm{i})k + (0.5555 + 0.0050\mathrm{i})(1 - (1.3609 + 0.0150\mathrm{i})k) = 0$$

$$1 + (0.6496 - 0.0010\mathrm{i})k - (14.6153 - 0.0293\mathrm{i})m - (1.5385 - 0.0038\mathrm{i})km = 0 \tag{3.65}$$

These equations were obtained using a symbolic code and are given here to four decimal places.

The solutions for k and m are found to be

$$k = 4.0000 - 0.0084\mathrm{i}$$
$$m = 0.1733 \tag{3.66}$$

There are in fact small imaginary parts present in both solutions though in the case of the added mass there is no imaginary part to four places of decimals. These imaginary components represent the damping necessary to place the assigned pole and zero exactly on the imaginary axis of the complex eigenvalue plane. In engineering applications, especially with steel structures that are lightly damped, poles occur close to but not exactly on the imaginary axis. Assigning a natural frequency exactly on the imaginary axis may result in a complex modification term. By neglecting the imaginary part of the modification, the pole will not exactly be assigned to desired location but very close by. Figure 3.9 shows the modified receptances with an antiresonance at 6 rad/s and a natural frequency at 3 rad/s as expected.

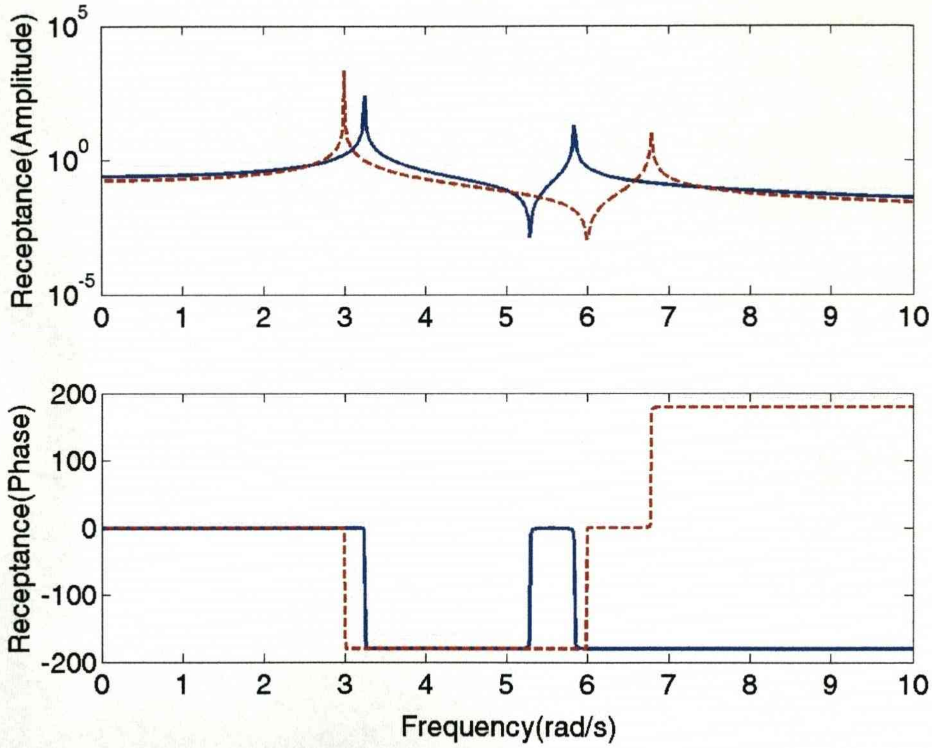

Figure 3.9. Original (full) and modified (dashed) receptances: assignment of the antiresonance at 6 rad/s and natural frequency at 3 rad/s

In the more general case of assigning several poles and zeros to a multi-dof system a set of complex non-linear multivariate polynomials in the modification parameters is revealed. Kyprianou *et al.* [K6] considered the inverse problem of assigning natural frequencies by an added mass connected by one or more springs. They used Grobner bases for the solution of such multivariate polynomials which assigned the desired natural frequencies and antiresonances. The same authors used an added beam as the modification to assign certain natural frequencies and antiresonances [K7]. A difficulty arises because realistic modifications such as an added beam or a large overhanging mass require the measurement of rotational receptances at the modification coordinate. In a companion paper [M13], they presented the so-called T-block approach to measure the rotational receptances. The method is extended to the X-block approach as will be described in chapter four to measure the rotational receptances from a Lynx helicopter tail-cone.

3.6 Modification by active control

The assignment of poles and zeros by adding structural elements such as point masses, springs or dampers restrict the system to satisfy certain properties such as symmetry, positive definiteness, reciprocity which is the characteristic of self-adjoint system. In addition passive modification requires the measurement of the rotational receptances at the connection points of the modification to the structure. In chapter four a technique to measure the rotational receptances is demonstrated. However it can be concluded that measuring the rotational receptances requires an expensive procedure and high level of specialist expertise. Also the number of eigenvalues to be assigned needs to be matched by the rank of modification. These are some disadvantages of physically modifying the structure; however the stability of the modified system is guaranteed due to the interlacing of poles and zeros in the complex plane. Unlike passive modifications, modifications in actively controlled systems are flexible and can be designed according to a variety control laws. The main concern in actively controlled systems is the stability of the modified systems. It will be seen in chapter five that active control allows rank-one modifications that cannot be achieved passively so that all the poles may be assigned to arbitrarily prescribed locations when the system is controllable. Wonham [W3] showed that poles of a system could be assigned by state feedback if the system was controllable. In state feedback the control forces are described by a linear combination of the states, i.e. the position and velocity of the various dofs. Another approach for the eigenvalue assignment problem is the output feedback control which does not require all the measurements of the states. The actuator force can be a function of the sensor output at the same point for the collocated control. Kautsky *et al.* [K3] developed numerical methods for determining well-conditioned solutions to the pole assignment problem using state feedback. Chu and Datta [C6] showed that the displacement and velocity feedback gain matrices contained strictly real terms when self-conjugate poles were assigned. Datta *et al.* [D5] developed a closed-form solution for the partial pole assignment problem, in which chosen eigenvalues were relocated and all the other eigenvalues were left unchanged.

Pole assignment is a fundamental problem in control, and can be solved by several methods. A well-known state-space method can be used in which the set of first order differential equations is considered.

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{b}\mathbf{u}(t) \tag{3.67}$$

where the matrix $\mathbf{A}$ represents the dynamic of the system, $\mathbf{u}(t)$ is the control force, and $\mathbf{b}$ is a vector that describes the distribution of the control force amongst the various dofs. Meirovitch developed the 'independent modal-space control' method in the book 'Dynamics and Control of Structures' [M6]. This approach allows in principle controlling one structural mode independently of the others. In practice, the control force must be applied in the physical coordinates, which means that sufficient actuators must be used to ensure that the selected mode is controllable while the others are unaffected by the control force. Meirovitch's analysis is presented in terms of the physical mass, damping and stiffness, **M**, **C**, **K**, matrices finally arranged in the form of first-order state-space equations (when general viscous damping is included).

$$\mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix},\quad \mathbf{q}=\begin{pmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{pmatrix} \tag{3.68}$$

Solving the first order system requires the knowledge of the matrix $\mathbf{A}$ and its complete set of eigenvalues and eigenvectors. In practice for structural vibratory systems since: (a) there is no systematic way to determine the damping matrix $\mathbf{C}$, and (b) vibrations of continuous structures are represented by matrices of large dimensions, therefore determining all eigenvalues and eigenvectors cannot be performed accurately by existing numerical algorithms. In most cases the **M**, **C**, **K** matrices are obtained by finite elements, and may include the representation of distributed piezo-electric actuators and sensors, as described for example by Lim *et al.* [L2]. Datta, Elhay and Ram [D13] used the **M**, **C**, **K** matrices and the eigenpairs associated with the poles intended to assign without changing the locations of all other poles. Instead of modifying the system parameters, an external control force $\mathbf{b}u(t)$, is applied to the system such that

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{b}u(t) \tag{3.69}$$

where $\mathbf{b}$ is a constant vector defining the position of applied forces and $u(t)$ is the control function. With a strategy of state feedback control, the function $u(t)$ forms a linear combination of the position and velocity of the various dofs, i.e.

$$u(t)=-(\mathbf{f}^T\dot{\mathbf{x}}+\mathbf{g}^T\mathbf{x}) \tag{3.70}$$

where $\mathbf{f}, \mathbf{g}$ are real constant vectors. Substituting Eq. (3.70) in Eq. (3.69) gives

$$\mathbf{M}\ddot{\mathbf{x}} + \left(\mathbf{C} + \mathbf{b}\mathbf{f}^T\right)\dot{\mathbf{x}} + \left(\mathbf{K} + \mathbf{b}\mathbf{g}^T\right)\mathbf{x} = \mathbf{0} \tag{3.71}$$

and hence the effect of the control is in modifying the damping and stiffness matrices by the non-symmetric rank one matrices $\mathbf{b}\mathbf{f}^T$ and $\mathbf{b}\mathbf{g}^T$. Structural modification, which is characterised by adding passive elements such as springs or dampers, modifies the system matrices in a symmetric manner, whereas active control allows non-symmetric modification of the form presented in Eq. (3.71). The consequence of the flexibility in applying the non-symmetric modification is that the entire dynamic of the system may be altered by active control in the sense that all poles of the system may be assigned to prescribe positions in the complex plane arbitrarily, provided that the system is controllable, i.e.

$$\operatorname{rank}\left(\left[\sigma_i^2\mathbf{M} + \sigma_i\mathbf{C} + \mathbf{K} \mid \mathbf{b}\right]\right) = n \tag{3.72}$$

for all eigenvalues σ_i, $i = 1,2,\ldots,2n$.

Another way of implementing the independent modal-space control of Meirovitch is to use modal test data, derived from measured receptances, as described by Stobener and Gaul [S8] in the active vibration control of a car body. In this case Rayleigh *ad hoc* (proportional) damping was considered. An alternative approach is to use the measured receptances directly without making any assumptions for damping as proposed by Ram and Mottershead [R6]. In their approach measured receptances from the original (open-loop) system could be used to assign all the poles of the closed loop system using just a single actuator. This research extends their approach to the output feedback control and the poles and zeros (antiresonances) are assigned using the measured receptances that are given by, $\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}$ without the need to know or evaluate the **M**, **C**, **K** matrices. Additionally, this method preserves the second-order nature of the physical system.

From the point of view of the structural dynamics it is preferable to work with the second-order matrix pencil [T1]. Redefining the second-order equations of motion into a first-order realisation destroys the desirable physical matrix properties of symmetry, definiteness and bandedness, and consequently the first-order state space model does not preserve any notion of the second-order nature of the system. The theory to define the control feedback gains for the single-input and multiple- input control force by the receptance method is presented in chapters six and seven.

3.7 Conclusion

In this chapter the theory of structural modification for rank one modification and general modification was presented. The effect of modification on the dynamic behaviour of the system was determined using the receptance method. The theory of Vincent Circle was discussed and it was shown that using Vincent Circle the vibration can be minimised but may not be exactly zero. The assignment of natural frequencies and antiresonances was demonstrated by numerical examples for both rank 1 modification and modification of higher orders. An introduction to the active control theory for the assignment of eigenvalues was presented and some advantageous of active control over passive modification were discussed.

Chapter 4

STRUCTURAL MODIFICATION OF A HELICOPTER TAILCONE

4.0 Introduction

This chapter presents the structural modification of a Lynx Mark 7 helicopter tailcone[1] [M15]. The full 6×6 receptance matrix of the tailcone is required to accomplish the structural modification. The rotational receptances are obtained with the aid of an X-block attachment to the tailcone. The X-block is represented by a finite element model and included in the formulation of multiple-input-multiple-output H_1 and H_2 estimators. Very good estimates of the rotational receptances are obtained comparing to the finite element results; the only significant difference is that the FE model is stiffer than the physical structure. A large overhanging mass, which represents the mass of the tail rotor gear box and the hub, is considered for the mass modification of the tailcone. The structural modification theory is used to determine the modified receptances. The importance of the rotational receptances and the effects of the weakly excited responses in determining the modified receptances are presented. It is also demonstrated, using the finite element model, that the use of a single X-block measurement can predict the first peak of the modified response accurately due to the coupling effects between the possible motions of the tail-cone.

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[1] The provision of the tail-cone by QinetiQ Ltd Farnborough is acknowledged. The experiments on the tail-cone were carried out in the BLADE Laboratory by kind permission of the University of Bristol.

Structural modification is a procedure which is used to predict the response of the modified structure from the initial response. There are some difficulties associated with structural modification problems such as measuring the rotational receptances which have a significant role in defining the modified response particularly at the connection points where the modification takes place. The lack of availability of the sensitive rotational accelerometers as well as the difficulty to apply a pure moment to the structure complicates the structural modification problem. Many techniques were suggested in the references of [M13] to estimate the rotational receptances which were based on mass additive technique, estimation techniques, laser Doppler vibrometer and rigid block attachment etc. A common approach such as the rigid block attachment carries certain assumptions and ill-conditioning in the resulting rotational receptances. Mottershead *et al.* [M13] developed a technique which includes the flexibility of the attachment in the formulation of the estimator. In [K7] the measured rotational receptances were used to assign the natural frequencies and antiresonances of a Γ-shaped structure by means of an added beam. The theory for estimating the rotational receptances of a T-block attachment is described in [M13]. The use of T-block enables the estimation of an in-plane 3×3 receptance matrix. However the cross rotational receptances can not be obtained from the T-block attachment.

Much attention nowadays is focused on inverse structural modification for assigning natural frequencies (poles) and antiresonances (zeros). The inverse structural modification has many applications in active control. For example, using the measured receptances in the structural modification theory, the poles and zeros of the modified structure can be assigned to desired locations. Mottershead and Ram [M14] considered the inverse problem of determining the passive modification to assign the selected eigenvalues (poles and zeros) to a dynamic system.

This chapter begins with explanation of estimating the rotational receptances for a Lynx Mark 7 helicopter tailcone using an X-block attachment. An overview of modification theory with its application to a helicopter tailcone for an overhanging mass modification is given. The finite element analysis is used to verify the initial and the modified receptances.

4.1 Estimation of rotational receptances

The theory for estimating the rotational receptances of a T-block attachment is described in [M13]. This device enables the estimation of an in-plane 3×3 receptance matrix at two linear coordinates and one rotational coordinate. It is not possible to determine a full 6×6 matrix using T-bocks alone. Use of the X-block, shown in Figure 4.1, however enables the estimation a 5×5 matrix of receptances, and by moving the X-block into different positions the full 6×6 matrix may be obtained. The local X-block coordinate system shown in the figure is used in the formulation of the multiple-input, multiple-output estimator of the 5×5 receptance matrix at each position of the X-block.

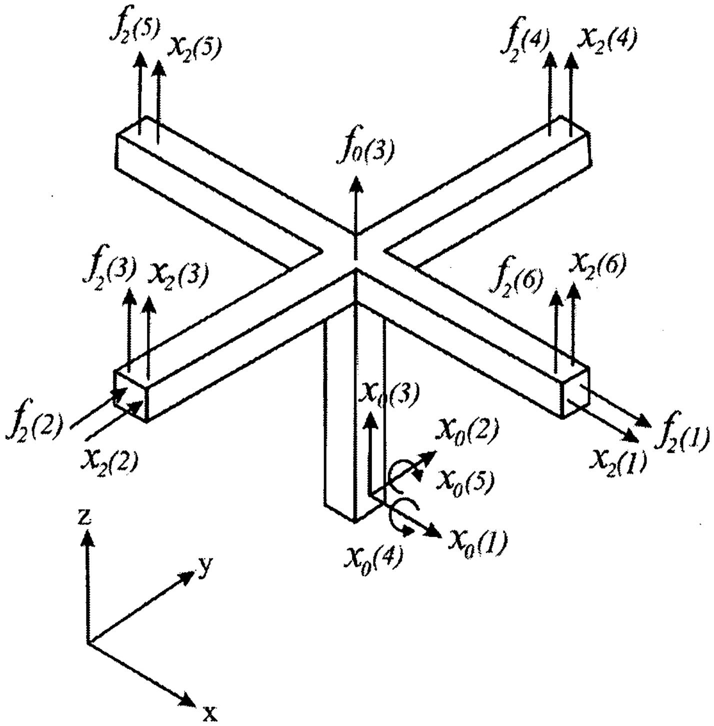


Figure 4.1. X-block forces and displacements

We begin, as in [M13], [M15] by writing the equation of motion of the system with the attached X-block,

$$\begin{bmatrix} \mathbf{Z}_{11}(\omega) & \mathbf{Z}_{10}(\omega) & \mathbf{0} \\ \mathbf{Z}_{01}(\omega) & \mathbf{Z}_{00}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_0 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_0 \\ \mathbf{f}_2 \end{pmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{Z}}_{00}(\omega) & \widetilde{\mathbf{Z}}_{02}(\omega) \\ \mathbf{0} & \widetilde{\mathbf{Z}}_{20}(\omega) & \widetilde{\mathbf{Z}}_{22}(\omega) \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_0 \\ \mathbf{x}_2 \end{pmatrix} \tag{4.1}$$

where the subscripts 0, 1, 2 denote the shared structure X-block connection-point coordinates, the coordinates of the structure (excluding the connection point) and the X-block coordinates (excluding the connection point) respectively. The submatrices $\mathbf{Z}_{11}(\omega), \mathbf{Z}_{01}(\omega), \mathbf{Z}_{00}(\omega)$ are unknown dynamic stiffnesses of the structure and $\widetilde{\mathbf{Z}}_{00}(\omega), \widetilde{\mathbf{Z}}_{20}(\omega), \widetilde{\mathbf{Z}}_{22}(\omega)$ are X-block dynamic stiffnesses determined from a finite element model.

Figure 4.1 shows the measured forces $\mathbf{f}_0(\omega)$ and $\mathbf{f}_2(\omega)$, the measured displacements $\mathbf{x}_2(\omega)$ (obtained from accelerometer measurements) and the unmeasured displacements $\mathbf{x}_0(\omega)$ defined in the local X-block coordinate system used in the following theory. The dimensions of the force and displacement vectors are as follows:

$$\mathbf{f}_0 \in \mathbb{C}^{5\times1}, \quad \mathbf{f}_2 \in \mathbb{C}^{6\times1}, \quad \mathbf{x}_2 \in \mathbb{C}^{6\times1}, \quad \mathbf{x}_0 \in \mathbb{C}^{5\times1} \tag{4.2}$$

where $\mathbb{C}$ denotes the set of complex numbers. The forces $\mathbf{f}_2(1)\ldots\mathbf{f}_2(6)$ and $\mathbf{f}_0(3)$ are applied consecutively in seven separate tests, so that for example, in the first test $\mathbf{f}_0$ is the null vector and only the first term in $\mathbf{f}_2$ is non-zero ($\mathbf{f}_0(i)=0, i=1,2,\ldots,5$ and $\mathbf{f}_2(i)=0, i=2,3,\ldots,6$).

Equation (4.1) may be separated into two parts,

$$\begin{bmatrix} \mathbf{Z}_{11}(\omega) & \mathbf{Z}_{10}(\omega) \\ \mathbf{Z}_{01}(\omega) & \mathbf{Z}_{00}(\omega) \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_0 - \left(\widetilde{\mathbf{Z}}_{00}(\omega)\mathbf{x}_0 + \widetilde{\mathbf{Z}}_{02}(\omega)\mathbf{x}_2\right) \end{pmatrix} \tag{4.3}$$

and,

$$\mathbf{f}_2 - \left(\widetilde{\mathbf{Z}}_{20}(\omega)\mathbf{x}_0 + \widetilde{\mathbf{Z}}_{22}(\omega)\mathbf{x}_2\right) = \mathbf{0} \tag{4.4}$$

The receptance matrix of the parent structure without the X-block is not measured directly but is defined as the inverse of the unknown dynamic stiffness matrix,

$$\begin{bmatrix} \mathbf{H}_{11}(\omega) & \mathbf{H}_{10}(\omega) \\ \mathbf{H}_{01}(\omega) & \mathbf{H}_{00}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{11}(\omega) & \mathbf{Z}_{10}(\omega) \\ \mathbf{Z}_{01}(\omega) & \mathbf{Z}_{00}(\omega) \end{bmatrix}^{-1} \tag{4.5}$$

Our objective is to determine the receptance submatrix $\mathbf{H}_{00}(\omega)$ and an expression in this term alone can be obtained from equation (4.3) after premultiplying by the matrix defined in equation (4.5). Then the second row of the resulting matrix equation gives,

$$\mathbf{x}_0 = \mathbf{H}_{00}(\omega)\left(\mathbf{f}_0 - \left(\widetilde{\mathbf{Z}}_{00}(\omega)\mathbf{x}_0 + \widetilde{\mathbf{Z}}_{02}(\omega)\mathbf{x}_2\right)\right) \tag{4.6}$$

Eliminating the unmeasured $\mathbf{x}_0$ by combining equations (4.6) and (4.4) leads to,

$$\begin{aligned}&\left(\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\widetilde{\mathbf{Z}}_{20}(\omega)\right)^{-1}\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\left(\mathbf{f}_2 - \widetilde{\mathbf{Z}}_{22}(\omega)\mathbf{x}_2\right) = \mathbf{H}_{00}(\omega)\\ &\times\left(\mathbf{f}_0 - \left(\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\widetilde{\mathbf{Z}}_{20}(\omega)\right)^{-1}\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\left(\mathbf{f}_2 - \widetilde{\mathbf{Z}}_{22}(\omega)\mathbf{x}_2\right) + \widetilde{\mathbf{Z}}_{02}(\omega)\mathbf{x}_2\right)\end{aligned} \tag{4.7}$$

where the finite-element dynamics stiffness submatrices have the following dimensions,

$$\widetilde{\mathbf{Z}}_{20}(\omega) \in \mathbb{C}^{6\times 5}, \quad \widetilde{\mathbf{Z}}_{00}(\omega) \in \mathbb{C}^{5\times 5}, \quad \widetilde{\mathbf{Z}}_{22}(\omega) \in \mathbb{C}^{6\times 6} \tag{4.8}$$

Equation (4.6) may be re-written in the simplified form,

$$\mathbf{R}(\omega)\mathbf{f}_2(\omega) + \mathbf{S}(\omega)\mathbf{x}_2(\omega) = \mathbf{H}_{00}(\omega)\left(\mathbf{f}_0(\omega) + \mathbf{T}(\omega)\mathbf{f}_2(\omega) + \mathbf{U}(\omega)\mathbf{x}_2(\omega)\right) \tag{4.9}$$

or,

$$\left[\mathbf{R}(\omega) \quad \mathbf{S}(\omega)\right]\begin{pmatrix}\mathbf{f}_2(\omega)\\ \mathbf{x}_2(\omega)\end{pmatrix} = \mathbf{H}_{00}\left[\mathbf{I} \quad \mathbf{T}(\omega) \quad \mathbf{U}(\omega)\right]\begin{pmatrix}\mathbf{f}_0(\omega)\\ \mathbf{f}_2(\omega)\\ \mathbf{x}_2(\omega)\end{pmatrix} \tag{4.10}$$

where,

$$\mathbf{R}(\omega) = \left(\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\widetilde{\mathbf{Z}}_{20}(\omega)\right)^{-1}\widetilde{\mathbf{Z}}_{20}^{*T}(\omega) \tag{4.11}$$

$$\mathbf{S}(\omega) = -\left(\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\widetilde{\mathbf{Z}}_{20}(\omega)\right)^{-1}\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\widetilde{\mathbf{Z}}_{22}(\omega) \tag{4.12}$$

$$\mathbf{T}(\omega) = -\widetilde{\mathbf{Z}}_{00}(\omega)\left(\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\widetilde{\mathbf{Z}}_{20}(\omega)\right)^{-1}\widetilde{\mathbf{Z}}_{20}^{*T}(\omega) \tag{4.13}$$

$$\mathbf{U}(\omega) = \widetilde{\mathbf{Z}}_{00}(\omega)\left(\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\widetilde{\mathbf{Z}}_{20}(\omega)\right)^{-1}\widetilde{\mathbf{Z}}_{20}^{*T}(\omega)\widetilde{\mathbf{Z}}_{22}(\omega) - \widetilde{\mathbf{Z}}_{02}(\omega) \tag{4.14}$$

are all matrices formed from the finite element model of the X-block. The formulation of the two estimators H_1 and H_2 will now be derived.

4.1.1 H_1 Estimator

Postmultiplying both sides of equation (4.10) by,

$$\begin{bmatrix}\mathbf{f}_0^{*T}(\omega) & \mathbf{f}_2^{*T}(\omega) & \mathbf{x}_2^{*T}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ \mathbf{T}^{*T}(\omega) \\ \mathbf{U}^{*T}(\omega)\end{bmatrix} \tag{4.15}$$

and taking n averages for each of the seven separate load cases leads to the expression,

$$\mathbf{B}(\omega) = \mathbf{H}_{00}(\omega)\mathbf{A}(\omega) \tag{4.16}$$

where,

$$\mathbf{A}(\omega) = \begin{bmatrix}\mathbf{I} & \mathbf{T}(\omega) & \mathbf{U}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{G}_{f_0 f_0} & \mathbf{G}_{f_0 f_2} & \mathbf{G}_{f_0 x_2} \\ \mathbf{G}_{f_2 f_0} & \mathbf{G}_{f_2 f_2} & \mathbf{G}_{f_2 x_2} \\ \mathbf{G}_{x_2 f_0} & \mathbf{G}_{x_2 f_2} & \mathbf{G}_{x_2 x_2}\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ \mathbf{T}^{*T}(\omega) \\ \mathbf{U}^{*T}(\omega)\end{bmatrix} \tag{4.17}$$

$$\mathbf{B}(\omega) = \begin{bmatrix}\mathbf{R}(\omega) & \mathbf{S}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{G}_{f_2 f_0}(\omega) & \mathbf{G}_{f_2 f_2}(\omega) & \mathbf{G}_{f_2 x_2}(\omega) \\ \mathbf{G}_{x_2 f_0}(\omega) & \mathbf{G}_{x_2 f_2}(\omega) & \mathbf{G}_{x_2 x_2}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ \mathbf{T}^{*T}(\omega) \\ \mathbf{U}^{*T}(\omega)\end{bmatrix} \tag{4.18}$$

and

$\mathbf{A}(\omega), \mathbf{B}(\omega) \in \mathbb{C}^{5\times 5}$.

The submatices, typically $\mathbf{G}_{f_0 x_2}(\omega)$, contain power spectral densities. For example,

$$\mathbf{G}_{f_0x_2}(\omega)=\sum_{1}^{7}\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} f_0^i(1)\,x_2^i(1) & f_0^i(1)\,x_2^i(2) & f_0^i(1)\,x_2^i(3) & f_0^i(1)\,x_2^i(4) & f_0^i(1)\,x_2^i(5) & f_0^i(1)\,x_2^i(6) \\ f_0^i(2)\,x_2^i(1) & f_0^i(2)\,x_2^i(2) & f_0^i(2)\,x_2^i(3) & f_0^i(2)\,x_2^i(4) & f_0^i(2)\,x_2^i(5) & f_0^i(2)\,x_2^i(6) \\ f_0^i(3)\,x_2^i(1) & f_0^i(3)\,x_2^i(2) & f_0^i(3)\,x_2^i(3) & f_0^i(3)\,x_2^i(4) & f_0^i(3)\,x_2^i(5) & f_0^i(3)\,x_2^i(6) \\ f_0^i(4)\,x_2^i(1) & f_0^i(4)\,x_2^i(2) & f_0^i(4)\,x_2^i(3) & f_0^i(4)\,x_2^i(4) & f_0^i(4)\,x_2^i(5) & f_0^i(4)\,x_2^i(6) \\ f_0^i(5)\,x_2^i(1) & f_0^i(5)\,x_2^i(2) & f_0^i(5)\,x_2^i(3) & f_0^i(5)\,x_2^i(4) & f_0^i(5)\,x_2^i(5) & f_0^i(5)\,x_2^i(6) \end{bmatrix} \tag{4.19}$$

where the subscript denoting the load case is omitted for clarity. Finally the H_1 estimate is given by,

$$\mathbf{H}_{00}(\omega)=\mathbf{B}^{-1}(\omega)\,\mathbf{A}(\omega) \tag{4.20}$$

4.1.2 H_2 Estimator

When equation (4.10) is postmultiplied on both sides by,

$$\begin{bmatrix}\mathbf{f}_2^{*T}(\omega) & \mathbf{x}_2^{*T}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{R}^{*T}(\omega)\\ \mathbf{S}^{*T}(\omega)\end{bmatrix} \tag{4.21}$$

then the following expression is obtained,

$$\mathbf{D}(\omega)=\mathbf{H}_{00}(\omega)\mathbf{C}(\omega) \tag{4.22}$$

where,

$$\mathbf{C}(\omega)=\begin{bmatrix}\mathbf{I} & \mathbf{T}(\omega) & \mathbf{U}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{G}_{f_0f_2}(\omega) & \mathbf{G}_{f_0x_2}(\omega)\\ \mathbf{G}_{f_2f_2}(\omega) & \mathbf{G}_{f_2x_2}(\omega)\\ \mathbf{G}_{x_2f_2}(\omega) & \mathbf{G}_{x_2x_2}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{R}^{*T}(\omega)\\ \mathbf{S}^{*T}(\omega)\end{bmatrix} \tag{4.23}$$

$$\mathbf{D}(\omega)=\begin{bmatrix}\mathbf{R}(\omega) & \mathbf{S}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{G}_{f_2f_2}(\omega) & \mathbf{G}_{f_2x_2}(\omega)\\ \mathbf{G}_{x_2f_2}(\omega) & \mathbf{G}_{x_2x_2}(\omega)\end{bmatrix}\begin{bmatrix}\mathbf{R}^{*T}(\omega)\\ \mathbf{S}^{*T}(\omega)\end{bmatrix} \tag{4.24}$$

and $\mathbf{C}(\omega), \mathbf{D}(\omega)\in\mathbb{C}^{5\times 5}$.

The H_2 estimate is then given by,

$$\mathbf{H}_{00}(\omega) = \mathbf{C}^{-1}(\omega)\,\mathbf{D}(\omega) \tag{4.25}$$

It can be seen that the difference between the two estimators is in the arrangements of the spectral density matrices and the matrices formed from the FE model of the X-block.

4.2 Finite element model of the X-block

The finite element model of the X-block consists of five Euler-Bernoulli beam elements as shown in Figure 4.2. The dynamic stiffness for the beam element is given by Narayanan [N2]. The X-block is made of a solid piece of mild steel. The dimension is given in Figure 4.3. An offset node is considered to model the joint between the four arms and the stem. The displacements at the connecting nodes are determined from rigid constraints. The four rotational coordinates at the tips of the arms and the two at the joint are unmeasured and must be eliminated from the model using Guyan reduction. Care should be taken in dealing with mass terms such the masses of the arms, which need to be added to the dynamic stiffness of the stem (at the joint) in both the local x and y coordinates. Similarly the mass of the stem needs to be added at axial coordinate $x_0(3)$.

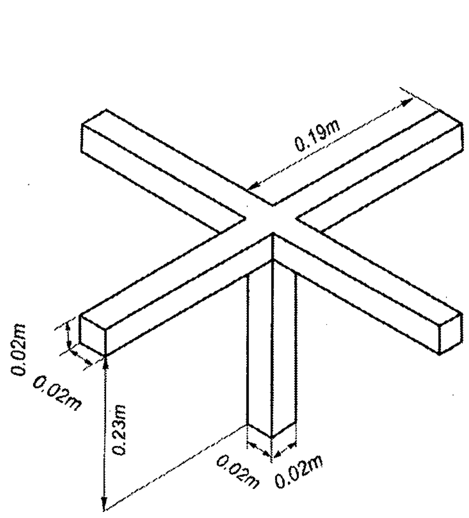


Figure 4.2. X-block dimensions

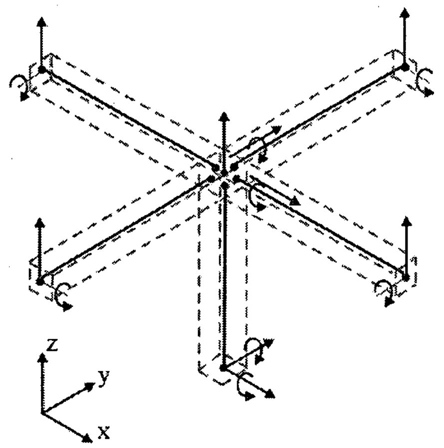


Figure 4.3. X-block finite element model

The first natural frequencies of the finite element model, fixed at the base of the stem, are two stem-bending modes at 61 Hz. Two corresponding frequencies at 58 Hz and 61 Hz were found in a hammer-excited modal test, with the base fixed to a rigid heavy metal block.

4.3 Helicopter tailcone

The Westland Mark 7 Lynx tailcone is shown together with the global axis system (used for the presentation of results) in Figure 4.4. It was detached at the transport joint and attached via an aluminium plate to a rigid wall. The tailcone is nearly, but not perfectly, symmetric about the X-Z plane. A large overhanging 76 kg mass, almost exactly the same mass as the tailcone itself and representative of the tail-rotor gearbox and hub, can be seen at the top left of the figure. The X-block attachment point, shown in Figure 4.5, did not coincide exactly with the attachment of the overhanging mass.

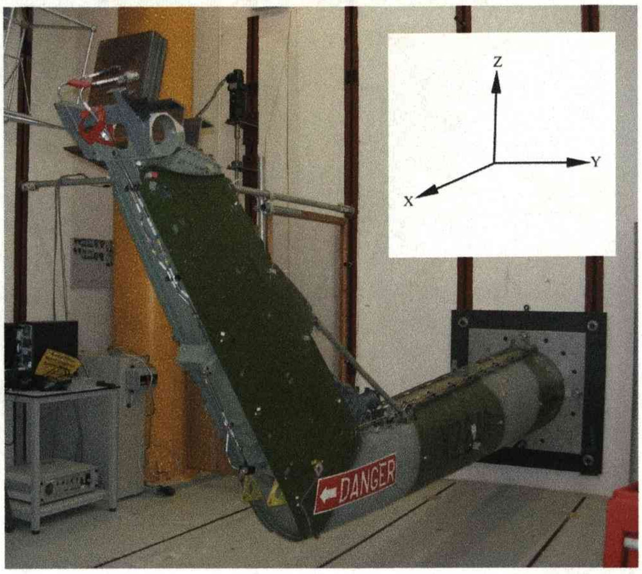


Figure 4.4. Tailcone global coordinate system

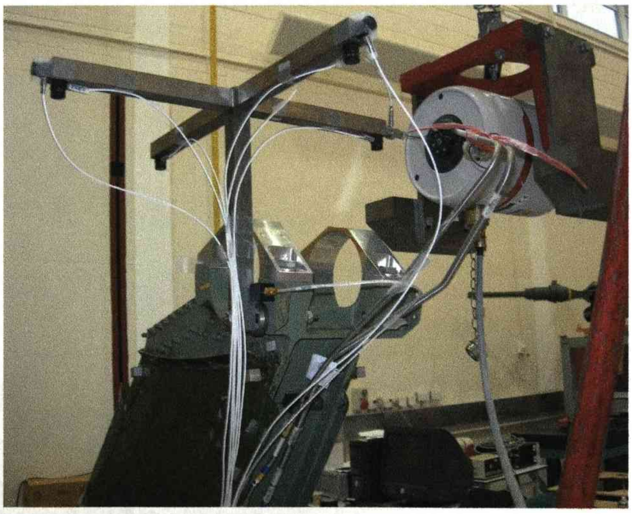


Figure 4.5. X-block attached to the tailcone

A preliminary hammer test was carried out on the baseline tailcone (without the added mass and also without the X-block). The first two vibration modes at 12.7 and 13.2 Hz showed coupling between the lateral (X-Z) and transverse (Y-Z) bending modes, the bending planes being almost perpendicular and at approximately 45° to the Y-Z axes. There was no discernable torsion about the X-axis. The third and Fourth modes at 46 and 58 Hz did however show rotation about the X-axis coupled with lateral and transverse bending. The large overhanging-mass modification was expected to introduce torsional coupling (and possibly yet more complicated coupling of displacement and rotation) even in the lowest frequency modes and therefore cross moment-rotation receptances were needed at the connection point in order to determine the dynamics of the added-mass system by the structural-modification theory. The finite element model of the tailcone (in MSC-NASTRAN) consisted of 2771 elements (1600 nodes) including 1508 CQUAD4 elements and 1139 CBEAM elements.

4.4 Estimated baseline-tailcone receptances

The estimated upper triangle of the 6×6 matrix of H_1 receptance from the X-block mounted in three mutually perpendicular orientations is given in Figures 4.6-4.8 together with the same matrix terms determined from the finite element model. There is some duplication of results when the X-block is mounted in the three different orientations, and of course certain configurations are best suited for the estimation of particular matrix terms. Of the duplicated estimates the least noisy ones were selected for presentation, though generally all of them were in quite reasonable agreement. The spectral densities, in equations (4.16)-(4.17), were determined directly from a modal test using random excitation over the range of 0-160 Hz and 512 spectral lines. An average of 500 measurements was taken with a 32% overlap. The receptances produced from the tail-cone finite element model were determined using modal damping at 0.1% of critical for all modes in the range 0-250 Hz.

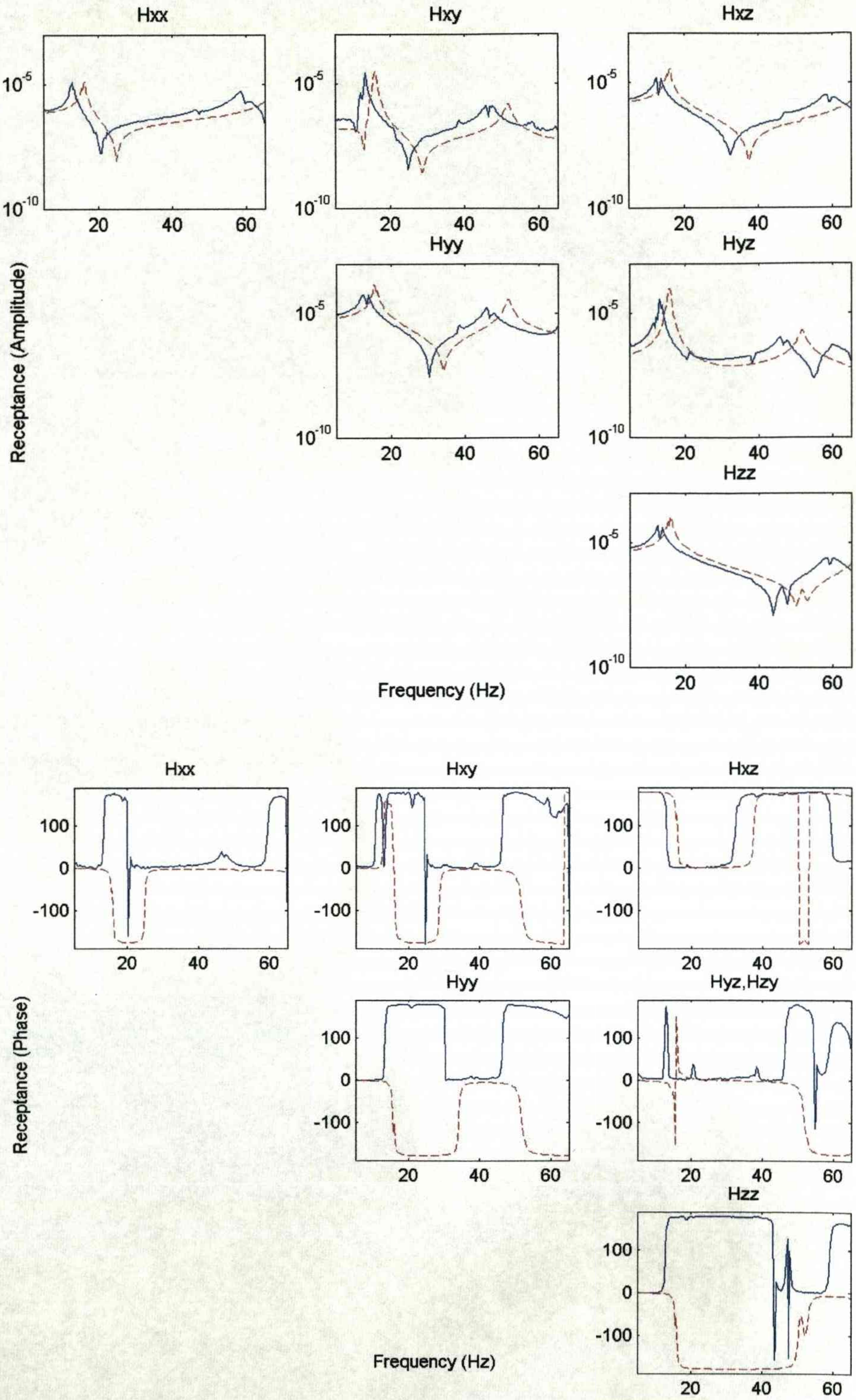


Figure 4.6. Upper left-hand receptance matrix (magnitude and phase)
solid line – estimated from X-block measurements
dashed line – finite element

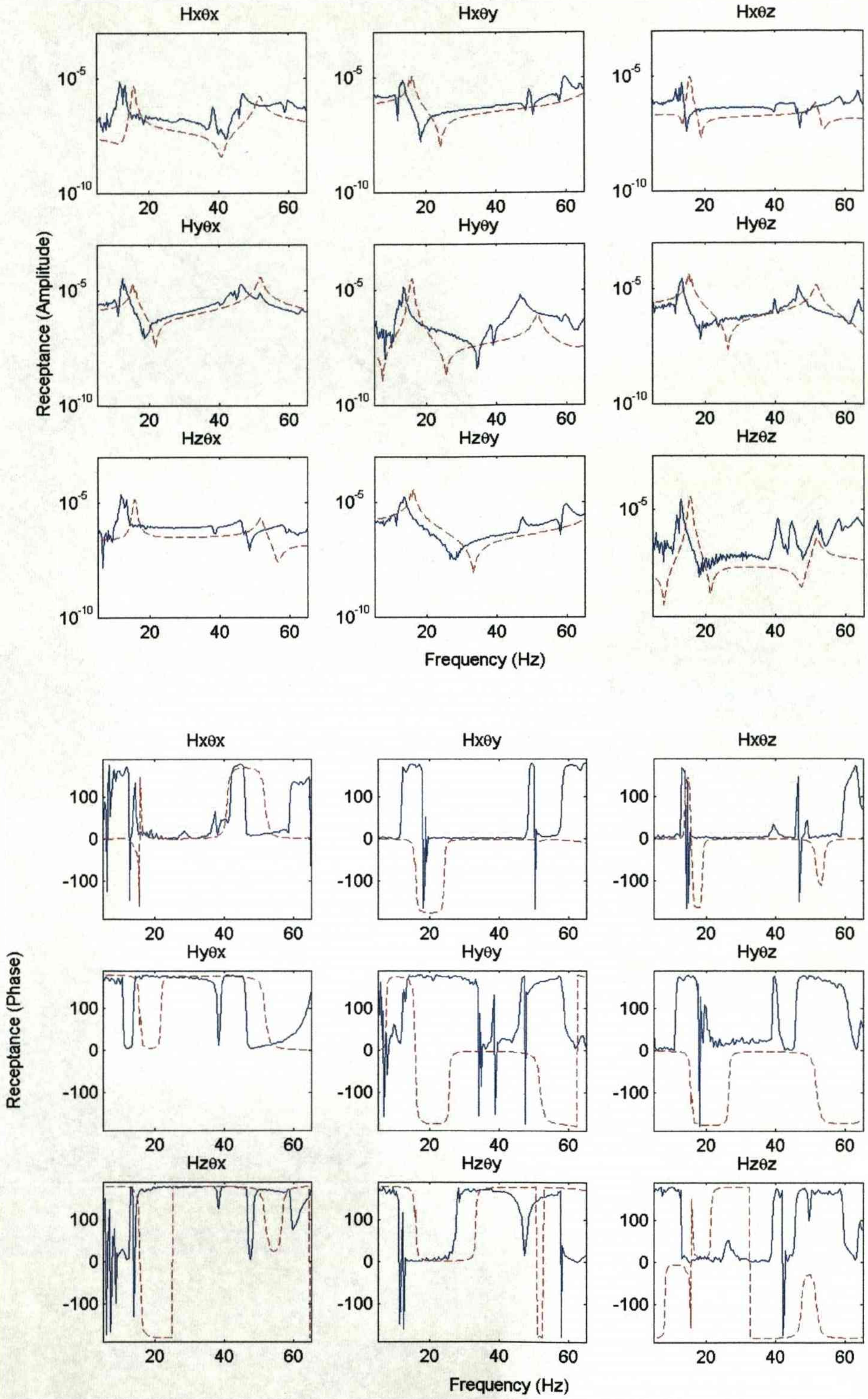


Figure 4.7. Upper right-hand receptance matrix (magnitude and phase)
solid line – estimated from X-block measurements
dashed line – finite element

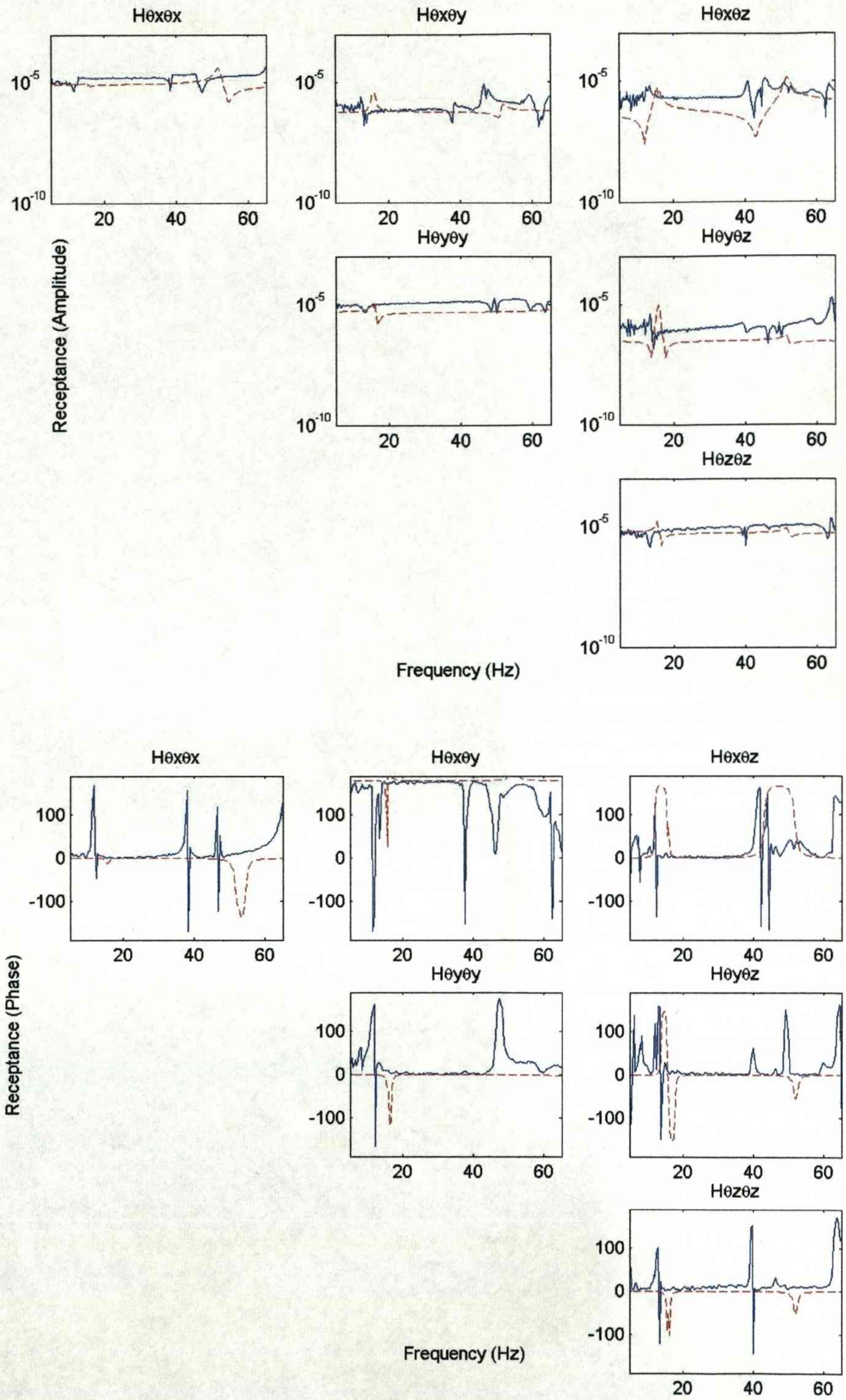


Figure 4.8. Lower right-hand receptance matrix (magnitude and phase)
solid line – estimated from X-block measurements
dashed line – finite element

It can be seen that the finite element model is stiffer than the physical system. Probably the most difficult terms to estimate are the terms on the diagonal of the upper right-hand sub-matrix in Figure 4.7, including the extremely difficult to obtain $H_{00}(1,4)$ representing the rotational response θ_x to the force f_x. It can be seen that the terms $H_{00}(1,4)$, $H_{00}(2,5)$ and $H_{00}(3,6)$ are amongst the most noisy estimates, yet still show good agreement with the finite element results. The rotational receptances, shown in Figure 4.8, appear to be quite different to the translational receptances we are used to seeing, being quite flat except for peaks and troughs over narrow frequency bands. Of course, it is important that not only the magnitudes but also the phases of the 6×6 matrix are accurately determined. The differences in the shape of the phase plots at the 180° resonance phase changes are due only to differences in the modal damping (at 0.1% of critical) applied in the finite element model from the damping of the real structure. The H_2 estimates appear to be generally of poorer quality than the H_1 estimates as shown for example in Figure 4.9.

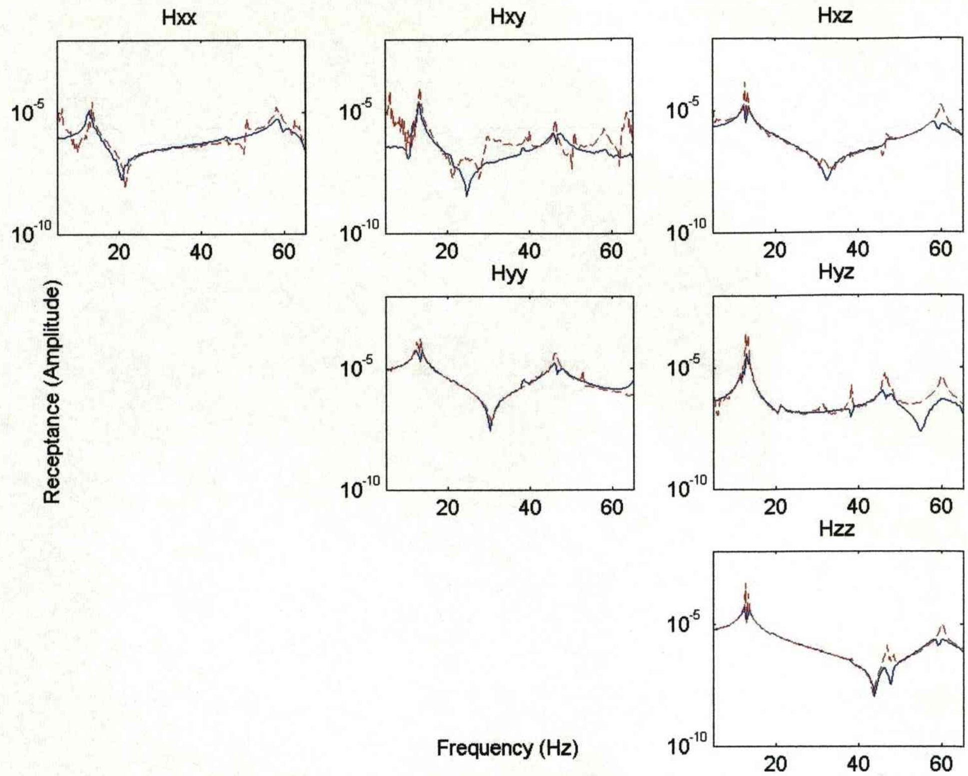


Figure 4.9. Upper left-hand receptance matrix (magnitude – m/N)

solid line – H_1 estimate
dashed line – H_2 estimate

4.5 Structural modification

The details of the structural-modification theory can be found in references [M6] and [M8], where it is explained how the receptance of a system $\mathbf{Z}(\omega)\mathbf{f}(\omega) = \mathbf{x}(\omega)$ modified by an added dynamic stiffness $\mathbf{\Delta Z}(\omega)$ may be written in the form,

$$\widetilde{\mathbf{H}}(\omega) = \left(\mathbf{I} + \mathbf{H}(\omega)\mathbf{\Delta Z}(\omega)\right)^{-1}\mathbf{H}(\omega) \tag{4.26}$$

where $\widetilde{\mathbf{H}}(\omega)$ represents the matrix of modified-system receptances and $\mathbf{H}(\omega)$ contains the receptances of the initial system.

4.5.1 *The overhanging mass modification*

The overhanging mass modification[2] and position vector $\vec{\mathbf{r}}$ of the centre of mass with respect to the connection point of the X-block is shown in Figure 4.10. The centre of gravity is denoted by '0' and the connection point by '1'. A solid model placed in position on the finite element mesh is shown in Figure 4.11. The modification consisted mainly of a solid piece of steel of dimensions 410×230×80 mm^3 and two channel section beams, each 125 mm × 65 mm (15 mm wall thickness) × 410 mm long. The attachment was via a 15 mm mild steel plate using ten 8 mm bolts connecting into the rigid tail-rotor gearbox mounting.

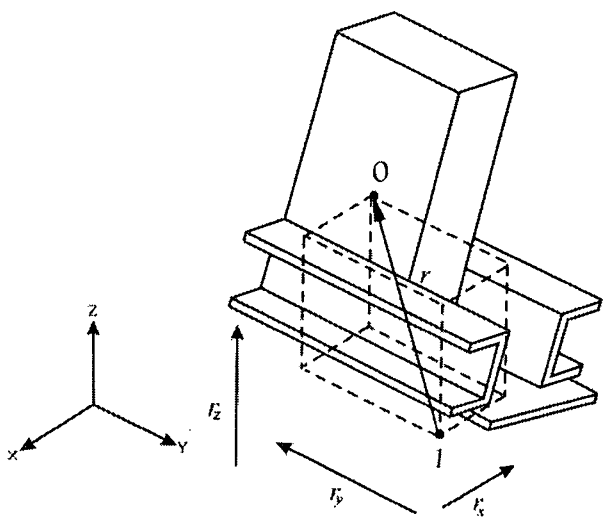


Figure 4.10. Mass modification and position vector.

[2] The provision of the overhanging mass by Westland Helicopters is acknowledged.

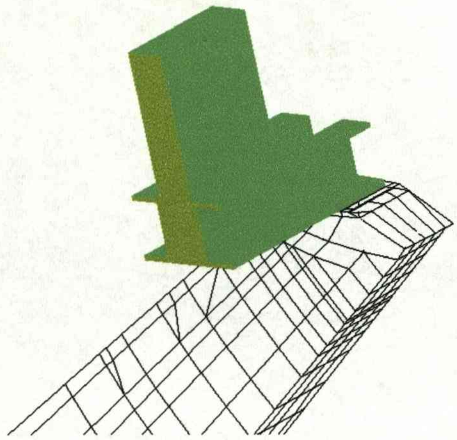


Figure 4.11. Solid model and FE mesh

For the purposes of applying the modification theory it was assumed that the modification itself was rigid and could therefore be represented by its total mass and the 3×3 inertia matrix at the centre of mass. This is described by the mass modification matrix,

$$\mathbf{\Delta M} = \begin{bmatrix} 75.6 & & & & & \\ & 75.6 & & & & \\ & & 75.6 & & & \\ & & & 1.844 & -0.044 & 0.184 \\ & & & -0.044 & 1.268 & 0.254 \\ & & & 0.184 & 0.254 & 0.832 \end{bmatrix} \tag{4.27}$$

The position vector that defines of the centre of mass with respect to the X-block connection point is given by,

$$\vec{\mathbf{r}} = -0.256\mathbf{i} - 0.213\mathbf{j} + 0.209\mathbf{k} \tag{4.28}$$

The terms in equation (4.28) also appear in the transformation matrix $\mathbf{T}$, so that the modification at the connection point is expressed in the form,

$$\mathbf{\Delta Z} = -\omega^2 \mathbf{T}^T \mathbf{\Delta M}\, \mathbf{T} \tag{4.29}$$

and applied according to equation (4.26). The transformation matrix is written as,

$$\mathbf{T}=\begin{bmatrix}1 & & & 0 & r_z & -r_y \\ & 1 & & -r_z & 0 & r_x \\ & & 1 & r_y & -r_x & 0 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{bmatrix}=\begin{bmatrix}1 & & & 0 & 0.209 & 0.213 \\ & 1 & & -0.209 & 0 & -0.256 \\ & & 1 & -0.213 & 0.256 & 0 \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{bmatrix} \quad (4.30)$$

This arrangement means that the mass modification is correctly located but connected by a single rigid link to the X-block connection point.

4.6 Modification results

A series of results are presented, beginning with a mass modification (without inertia) applied at the X-block connection point. Then the mass and inertia modification at the connection point ($\vec{\mathbf{r}}=\mathbf{0}$) is carried out. Finally, the complete modification including the full offset mass and inertia with position vector defined in equation (4.28) is considered.

4.6.1 Mass modification without inertia ($\vec{\mathbf{r}}=\mathbf{0}$)

The results of the mass-only modification are shown on Figure 4.12. The two sets of results, shown by the solid and dashed lines, are obtained by using the 6×6 receptance matrices shown in Figures 4.6-4.8. The receptances given by the solid lines are determined by applying equation (4.26) to the measurements (solid lines) in Figures 4.6-4.8. Correspondingly the dashed-line receptances shown in Figure 4.12 are obtained by applying equation (4.26) to the finite-element (dashed-line) receptances in Figures 4.6-4.8. The upper left-hand sub-matrix is sufficient to show that the main features of the two sets of receptances are very similar. As with the initial-system receptances the finite element model is found to be stiffer than the physical tailcone.

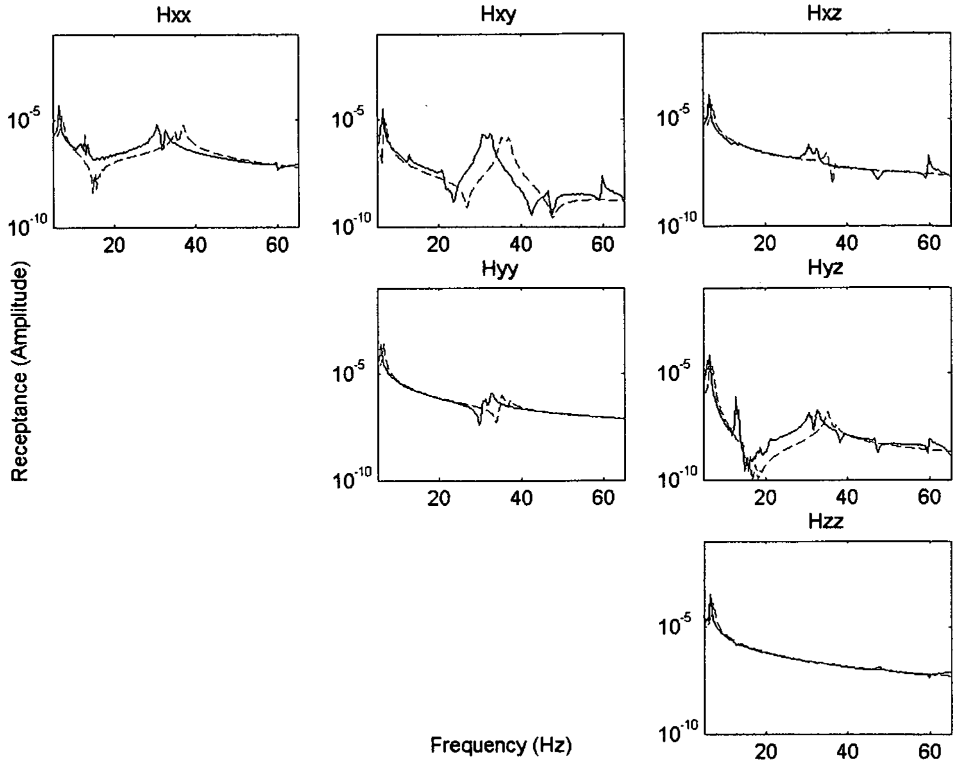


Figure 4.12. Receptances (m/N) - mass-only modification
solid line – experimental
dashed line – finite element

4.6.2 Mass modification with inertia ($\vec{\mathbf{r}} = \mathbf{0}$)

It is seen from Figure 4.13 that the effect of the inertia is significant. The prominent double measured peak at around 30 Hz in receptance h_{xy} shown in Figure 4.12 has changed shape completely in Figure 4.13. Note though, that the shapes of the diagonal-term receptances have not changed very much, except for a fairly modest reduction in the natural frequencies. This is because this modification with $\vec{\mathbf{r}} = \mathbf{0}$ does not introduce the strong coupling between the different motions of the tailcone that appear when the mass modification is overhanging.

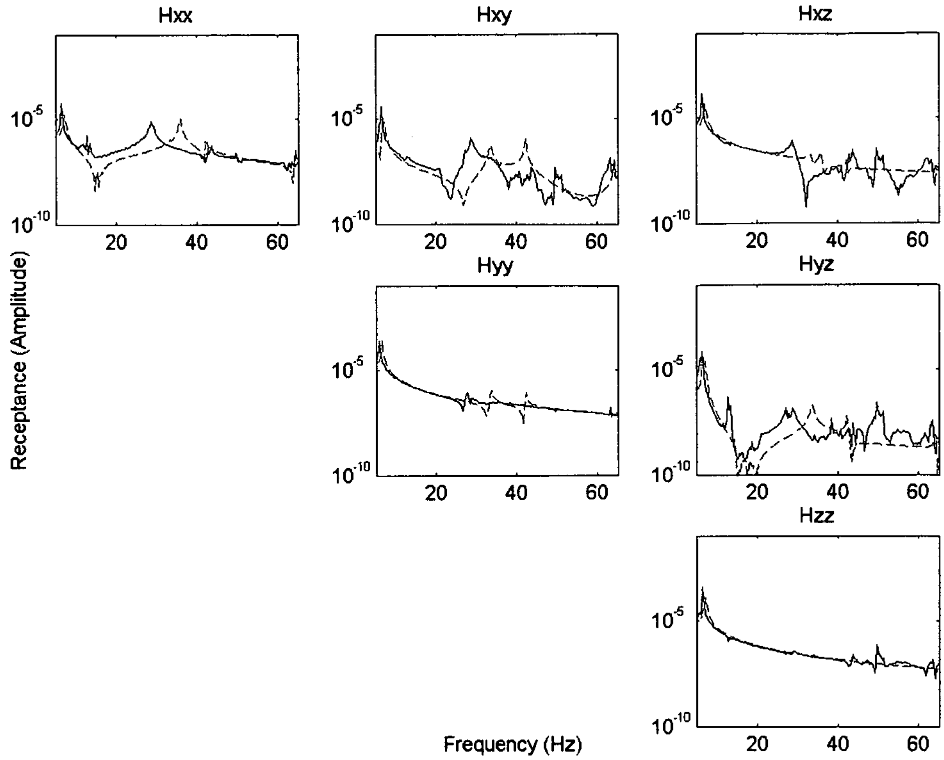


Figure 4.13. Receptances (m/N) – mass and inertia modification ($\vec{\mathbf{r}} = \mathbf{0}$)
solid line – experimental
dashed line – finite element

4.6.3 Full offset mass and inertia modification

From the result shown in Figure 4.14 it is apparent that the effect of the offset (overhang) is indeed very significant. The shapes of the receptances and the natural frequencies are all quite different to those shown in Figure 4.13. There remains, however, good agreement in the general shape of the solid-line and dashed-line receptances shown in Figure 4.14, although the noise that was apparent on the experimental solid lines in Figure 4.13 has increased in magnitude on the solid-line receptances in Figure 4.14. This is because the dynamic behaviour of modified system is now very different to that of the initial system, as can be seen by comparing Figure 4.14 to Figure 4.6. The modified receptances were obtained using equation (4.26), which requires the full 6×6 receptance matrix of the initial system, including the difficult to measure receptances such as the diagonal terms of the submatrix shown in Figure 4.7, which were noticed previously to be noisy. This noise on the initial-system receptances is amplified in the data processing of equation (4.26) because the overhanging mass now causes the coupling of different tailcone motions that were virtually uncoupled in the initial system.

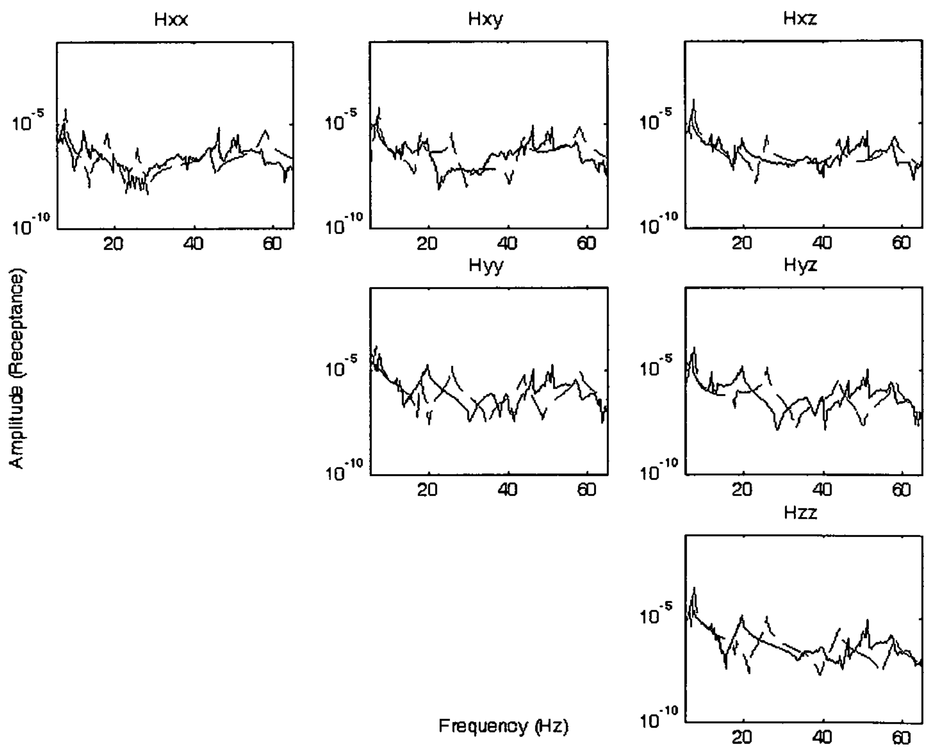


Figure 4.14. Receptances (m/N) – full overhanging mass and inertia modification

solid line – experimental

dashed line – finite element

4.7 Conclusion

The structural modification of Lynx Mark 7 helicopter tailcone is presented in this chapter. The full 6×6 matrix of receptances is estimated using an X-block attachment. The X-block is represented by a finite element model and included in the formulation of the H_1 and H_2 estimators. The estimated rotational receptances show very good agreement with results from a finite element model, the only significant difference being that the finite element model is stiffer than the real structure. The modification, in the form of a large overhanging mass (representative of the tail-rotor gearbox and hub and of about the same mass as the baseline tailcone) is considered for the structural modification. It is shown that the modification has the effect of coupling all of the initial-system receptances. This has the effect of amplifying the noise of the weakly excited initial-system responses. However, application of the modification theory to both measured and FE results with a rigid connection produce similar shapes for the modified receptances. It can be concluded that the rotational receptances have significant effects in structural modification procedure. However, the measurement of rotational receptances requires high level of specialist expertise and therefore can be a main drawback in the application of passive modification. In the next chapter, the inverse problem of pole placement using active control based on the system matrices $\mathbf{M}, \mathbf{C}, \mathbf{K}$ will be discussed.

Chapter 5

ACTIVE CONTROL FOR POLE PLACEMENT USING STATE FEEDBACK

5.0 Introduction

In this chapter the pole placement problem using state feedback control based on the conventional methods using system matrices $\mathbf{M}, \mathbf{C}, \mathbf{K}$ is reviewed. The purpose of including this chapter is to understand the usual pole placement technique and also the difficulties associated with it. In later chapters a new approach, known as the *receptance method* is introduced which does not require the $\mathbf{M}, \mathbf{C}, \mathbf{K}$ matrices, but uses vibration measurements instead.

A brief summary of the recently derived solutions by Datta *et al.* [D3], [D4] for the partial pole placement is presented. The linear state feedback control is considered for the system modelled by a set of second order differential equations. Orthogonality relations are used to derive the solution of the partial pole assignment problem. The pole assignment is carried out using single-input and multiple-input state feedback control. The non-iterative algorithm defines a closed-form solution to the partial pole assignment problem. For the case of single-input feedback control a unique solution is derived, however for the case of multiple-input control a family of solutions may be obtained. The methods are illustrated with numerical examples.

5.1 Partial pole assignment

The problem of assigning a subset of eigenvalues by feedback control, while the remaining eigenvalues are invariant, is called the *partial* *eigenvalue assignment.* The most important application of partial eigenvalue assignment is in the relocation of those eigenvalues not in the desired region of the complex plane or those which lead to large amplitude vibrations. The problem requires us to find the control feedback gains so that certain closed-loop eigenvalues are prescribed and the remaining closed-loop eigenvalues are the same as the eigenvalues of the open-loop system. The partial eigenvalue problem can be considered for both single-input and multiple-input control. An obvious approach for the above problem is to convert the second order differential equations into the first order form and then apply one of the well-established techniques for the eigenvalue assignment of a first order system [D6], or more appropriately the projection and deflation method of Saad [S1]. However, from the point of view of the structural dynamics it is preferable to work with the second-order matrix pencil. Redefining the second-order equations of motion into a first-order realisation destroys the desirable physical matrix properties of symmetry, definiteness, sparsity and bandedness. Furthermore, the first-order state space model does not preserve any notion of the second-order nature of the system.

Another approach, the independent modal space control, has in principle many advantages but is difficult to implement in practice [M6]. The basic idea is to decouple the problem into a set of n independent equations and solve them separately. However, decoupling the right hand sides of the modal equations corresponding to the control vector requires a sufficient number of actuators to ensure that the selected mode is controllable while the others are unaffected by the control force.

In this chapter a method proposed by Datta *et al.* [D2]-[D7] is presented for the partial eigenvalue assignment which uses the $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ matrices from the second order form of equations and also the eigenpairs associated with the open loop system. Three important orthogonality relations are stated for the eigenvectors of the open loop system and are used to derive the solution for the partial eigenvalue assignment. In this approach theoretically no spillover occurs because the eigenvalues and eigenvectors that are not required to be altered are unaffected by the feedback control. This is different to what generally occurs in practice when the feedback control forces excite the unassigned poles, resulting in spillover. Furthermore, the method may contain modelling errors and assumptions since the **M**, **C**, and **K** matrices are usually formed by finite element methods. An alternative approach introduced by Mottershead and Ram [R6] has significant advantages over the conventional state-space approach and the second order method presented in this chapter, since it does not require the **M**, **C**, **K** matrices and is solely based on the measured receptances. The method is fully described in chapter six.

The dynamic equation of motion of a second order system is considered as;

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{5.1}$$

Resonances may occur when one or more eigenvalues of the characteristic equation become equal or close to the frequencies of the external force. To avoid such unwanted oscillations a control force

$$\mathbf{f}(t) = \mathbf{B}\mathbf{u}(t) \tag{5.2}$$

is applied, where $\mathbf{B} \in \Re^{n \times m}$ is the control force distribution matrix, and $\mathbf{u}(t) \in \Re^{m \times 1}$ is a time-dependent control force. The control force $\mathbf{u}(t)$ is formed from a linear combination of the states of the system.

$$\mathbf{u}(t) = -\mathbf{F}^T\dot{\mathbf{x}}(t) - \mathbf{G}^T\mathbf{x}(t) \tag{5.3}$$

where $\mathbf{F}, \mathbf{G} \in \Re^{m \times n}$. Therefore, the system (5.1) becomes;

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \left(\mathbf{C} + \mathbf{B}\mathbf{F}^{\mathrm{T}}\right)\dot{\mathbf{x}}(t) + \left(\mathbf{K} + \mathbf{B}\mathbf{G}^{\mathrm{T}}\right)\mathbf{x}(t) = \mathbf{0} \tag{5.4}$$

The problem is then to choose the matrices $\mathbf{F}$ and $\mathbf{G}$ such that the eigenvalues of the closed-loop characteristic equation satisfy Eq. (5.5).

$$\lambda^2\mathbf{M} + \lambda\left(\mathbf{C} + \mathbf{B}\mathbf{F}^T\right) + \left(\mathbf{K} + \mathbf{B}\mathbf{G}^T\right) = 0 \tag{5.5}$$

Suppose that we wish to change the partial set of eigenvalues σ_i, $i = 1,2,\ldots,p$, $p \leq 2n$ to a prescribed set λ_i, $i = 1,2,\ldots,p$, while leaving the other eigenvalues σ_i, $i = p+1, p+2,\ldots,2n$, unchanged. This leads to the following partial eigenvalue assignment problem.

Given

1. $\mathbf{M},\mathbf{C},\mathbf{K} \in \Re^{n\times n}$, $\mathbf{M} = \mathbf{M}^T > 0$, $\mathbf{C} = \mathbf{C}^T$, $\mathbf{K} = \mathbf{K}^T$.

2. Control matrix $\mathbf{B} \in \Re^{n\times m}$, $m \leq n$.

3. The self-conjugate spectral matrix of the open-loop system $\mathbf{\Sigma}_1 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots \sigma_p)$ and its associated eigenvector $\mathbf{V}_1 = \lfloor \mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_p \rfloor$.

4. The self-conjugate matrix of the prescribed spectra $\mathbf{\Lambda}_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_p)$.

Find real feedback matrices $\mathbf{F}$ and $\mathbf{G}$ such that the eigenvalues of the closed-loop characteristic equation are $\{\lambda_1, \lambda_2, \ldots \lambda_p, \sigma_{p+1}, \ldots, \sigma_{2n}\}$ where the closed-loop prescribed eigenvalues are denoted by $\{\lambda_1, \lambda_2, \ldots \lambda_p\}$.

Let $\mathbf{V} \in \mathbb{C}^{n\times 2n}$ and $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots \sigma_{2n}) \in \mathbb{C}^{2n\times 2n}$ be respectively the eigenvector and the eigenvalue matrix of the quadratic pencil $\mathbf{M}\mathbf{V}\mathbf{\Sigma}^2 + \mathbf{C}\mathbf{V}\mathbf{\Sigma} + \mathbf{K}\mathbf{V}$.

The three orthogonality relations are defined as [D3];

$$\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{M}\mathbf{V}\boldsymbol{\Sigma} - \mathbf{V}^T\mathbf{K}\mathbf{V} = \mathbf{D}_1 \tag{5.6}$$

$$\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{C}\mathbf{V}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{V}^T\mathbf{K}\mathbf{V} + \mathbf{V}^T\mathbf{K}\mathbf{V}\boldsymbol{\Sigma} = \mathbf{D}_2 \tag{5.7}$$

$$\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{M}\mathbf{V} + \mathbf{V}^T\mathbf{M}\mathbf{V}\boldsymbol{\Sigma} + \mathbf{V}^T\mathbf{C}\mathbf{V} = \mathbf{D}_3 \tag{5.8}$$

where $\mathbf{D}_1$, $\mathbf{D}_2$ and $\mathbf{D}_3$ are the diagonal matrices. Furthermore,

$$\mathbf{D}_1 = \mathbf{D}_3\boldsymbol{\Sigma} \tag{5.9}$$

$$\mathbf{D}_2 = -\mathbf{D}_1\boldsymbol{\Sigma} \tag{5.10}$$

$$\mathbf{D}_2 = -\mathbf{D}_3\boldsymbol{\Sigma}^2 \tag{5.11}$$

The first orthogonality relation Eq. (5.6) is used in the derivation of the partial pole assignment solution.

5.2 Single-input case

In the single-input case, an external control force $\mathbf{b}u(t)$, is applied to the system such that

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{b}u(t) \tag{5.12}$$

where $\mathbf{b} \in \Re^{n\times 1}$ defines the position of applied forces and $u(t)$ is the control function. With a strategy of state feedback control, the function $u(t)$ forms a linear combination of the position and velocity of the various dofs, i.e.

$$u(t) = -\mathbf{f}^T\dot{\mathbf{x}} - \mathbf{g}^T\mathbf{x} \tag{5.13}$$

where $\mathbf{f}, \mathbf{g} \in \Re^{n\times 1}$. Substituting Eq. (5.13) into Eq. (5.12) leads to;

$$\mathbf{M}\ddot{\mathbf{x}} + \left(\mathbf{C} + \mathbf{b}\mathbf{f}^T\right)\dot{\mathbf{x}} + \left(\mathbf{K} + \mathbf{b}\mathbf{g}^T\right)\mathbf{x} = \mathbf{0} \tag{5.14}$$

It is well known that the system (5.14) is completely controllable if and only if

$$\operatorname{rank}\left\{\sigma^2\mathbf{M}+\sigma\mathbf{C}+\mathbf{K},\mathbf{b}\right\}=n \tag{5.15}$$

for every eigenvalue σ_i of the quadratic pencil $\sigma_i^{\,2}\mathbf{M}+\sigma_i\mathbf{C}+\mathbf{K}$. Complete controllability is a necessary and sufficient condition for the existence of $\mathbf{f}$ and $\mathbf{g}$ such that the closed-loop pencil has its prescribed eigenvalues. If the system is partially controllable i.e. if

$$\operatorname{rank}\left\{\sigma^2\mathbf{M}+\sigma\mathbf{C}+\mathbf{K},\mathbf{b}\right\}=n \tag{5.16}$$

only for p of the eigenvalues, $p<n$, then only those eigenvalues can be arbitrarily assigned by an appropriate choice of $\mathbf{f}$ and $\mathbf{g}$.

It will be shown by application that the vectors $\mathbf{f}$ and $\mathbf{g}$ are given by

$$\mathbf{f}=-\mathbf{M}\mathbf{V}_1\mathbf{\Sigma}_1\mathbf{p} \tag{5.17}$$

and

$$\mathbf{g}=\mathbf{K}\mathbf{V}_1\mathbf{p} \tag{5.18}$$

where $\mathbf{\Sigma}_1=\operatorname{diag}(\sigma_1,\sigma_2,\ldots\sigma_p)$ and $\mathbf{V}_1=\lfloor\mathbf{v}_1\ \mathbf{v}_2\ldots\mathbf{v}_p\ \rfloor$ are the eigenvalues and eigenvectors of the open-loop system and $\mathbf{p}=(p_1\ p_2\ \cdots\ p_p)^T$ is defined by its elements

$$p_j=\frac{1}{\mathbf{b}^T\mathbf{v}_j}\frac{\lambda_j-\sigma_j}{\sigma_j}\prod_{\substack{i=1\\i\neq j}}^{p}\frac{\lambda_i-\sigma_j}{\sigma_i-\sigma_j},\quad j=1,2,\ldots p \tag{5.19}$$

We demonstrate that with $\mathbf{f}$ and $\mathbf{g}$ given by Eqs. (5.17) and (5.18) the eigenvalues σ_j, $j=p+1,p+2,\ldots,2n$, remain invariant by the feedback control. Moreover, the corresponding eigenvectors $\mathbf{v}_j$, $j = p+1, p+2, \ldots, 2n$, are not affected by the control either. So all we need to prove is

$$\mathbf{M}\mathbf{V}_2\boldsymbol{\Sigma}_2^2 + \left(\mathbf{C} + \mathbf{b}\mathbf{f}^T\right)\mathbf{V}_2\boldsymbol{\Sigma}_2 + \left(\mathbf{K} + \mathbf{b}\mathbf{g}^T\right)\mathbf{V}_2 = \mathbf{0} \tag{5.20}$$

where, $\boldsymbol{\Sigma}_2 = \mathrm{diag}\left(\sigma_{p+1}, \sigma_{p+2}, \ldots \sigma_{2n}\right)$ (5.21)

and its associated eigenvector matrix

$$\mathbf{V}_2 = \left[\mathbf{v}_{p+1}\ \mathbf{v}_{p+2} \cdots \mathbf{v}_{2n}\right] \tag{5.22}$$

Indeed, substituting Eqs. (5-17) and (5-18) in the left-hand side of Eq. (5-20) gives,

$$\mathbf{M}\mathbf{V}_2\boldsymbol{\Sigma}_2^2 + \left(\mathbf{C} - \mathbf{b}\mathbf{p}^T\boldsymbol{\Sigma}_1\mathbf{V}_1^T\mathbf{M}\right)\mathbf{V}_2\boldsymbol{\Sigma}_2 + \left(\mathbf{K} + \mathbf{b}\mathbf{p}^T\mathbf{V}_1^T\mathbf{K}\right)\mathbf{V}_2 = \mathbf{0} \tag{5.23}$$

which simplifies to

$$-\mathbf{b}\mathbf{p}^T\left(\boldsymbol{\Sigma}_1\mathbf{V}_1^T\mathbf{M}\mathbf{V}_2\boldsymbol{\Sigma}_2 - \mathbf{V}_1^T\mathbf{K}\mathbf{V}_2\right) = \mathbf{0} \tag{5.24}$$

Since we know already that,

$$\mathbf{M}\mathbf{V}_2\boldsymbol{\Sigma}_2^2 + \mathbf{C}\mathbf{V}_2\boldsymbol{\Sigma}_2 + \mathbf{K}\mathbf{V}_2 = \mathbf{0} \tag{5.25}$$

and $\boldsymbol{\Sigma}_2$ and $\mathbf{V}_2$ are the eigenpairs of the open-loop system.

Therefore

$$\boldsymbol{\Sigma}_1\mathbf{V}_1^T\mathbf{M}\mathbf{V}_2\boldsymbol{\Sigma}_2 - \mathbf{V}_1^T\mathbf{K}\mathbf{V}_2 = \mathbf{0} \tag{5.26}$$

It thus follows that the orthogonality relation (5.6) implies through Eq. (5.23) that for each vector $\mathbf{p}$ the state feedback (5.13) in conjunction with Eqs. (5.17) and (5.18) has no influence on the eigenvalues σ_j, $j = p+1, p+2, \ldots, 2n$, and their corresponding eigenvectors. It is shown by Datta *et al.* [D6] that choosing $\mathbf{p}$ according to the formula (5.19) shifts the eigenvalues σ_i, $i=1,2,\ldots,p$, to the required set λ_i, $i=1,2,\ldots,p$.

We remark that if the sets of $\sigma_i, i=1,2,\ldots,p$, and λ_i, $i=1,2,\ldots,p$ are both self conjugate then $\mathbf{f}$ and $\mathbf{g}$ are real vectors and hence the control can be realised. It also follows from Eq. (5.19) that the control is finite if

(a) the vector $\mathbf{b}$ is not orthogonal to any one of the eigenvectors $\mathbf{v}_i, i=1,2,\ldots,p$.

(b) the eigenvalues to be shifted, σ_i, $i=1,2,\ldots,p$, are distinct, and in complex conjugate pairs.

(c) $\sigma_i \neq 0$ for the shifted eigenvalues, $i=1,2,\ldots,p$.

Example 5.1.

A three degree-of-freedom system is considered as shown in Fig.5.1.

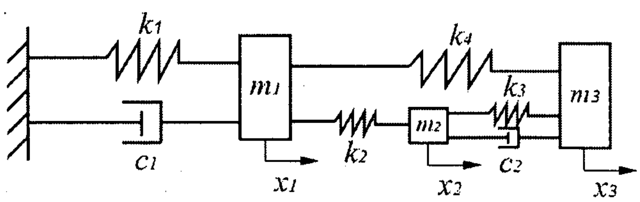


Figure 5.1. Three degree-of-freedom system

The mass, damping and stiffness matrices are given by;

$$\mathbf{M}=\begin{bmatrix}2&0&0\\0&1&0\\0&0&3\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}0.1&0&0\\0&0.1&-0.1\\0&-0.1&0.1\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}6&-2&-1\\-2&4&-2\\-1&-2&3\end{bmatrix}$$

The six eigenvalues of this system are:

$$\sigma_{1,2} = -0.0033 \pm 0.5513i$$
$$\sigma_{3,4} = -0.0374 \pm 1.6016i$$
$$\sigma_{5,6} = -0.0509 \pm 2.2638i$$

Suppose that, by using the state feedback control, we wish to change the eigenvalues $\sigma_{1,2}$ to the new values $\lambda_{1,2} = -0.005 \pm 0.7i$, while leaving the other four eigenvalues unchanged. We denote the closed-loop eigenvalues by λ.

Here

$$\mathbf{b} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$$

and

$$\mathbf{\Sigma}_1 = \begin{bmatrix} -0.0033 + 0.5513i & 0 \\ 0 & -0.0033 - 0.5513i \end{bmatrix}$$

The eigenvectors corresponding to λ_1 and λ_2 are:

$$\mathbf{v}_1 = \begin{pmatrix} 0.4831 - 0.0036i \\ 0.8026 + 0.0018i \\ 1.0000 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 0.4831 + 0.0036i \\ 0.8026 - 0.0018i \\ 1.0000 \end{pmatrix}$$

Hence,

$$\mathbf{V}_1 = [\mathbf{v}_1 \quad \mathbf{v}_2]$$

and Eq. (5.14) gives

$$p_1 = \frac{1}{\mathbf{b}^T\mathbf{v}_1}\frac{\lambda_1-\sigma_1}{\sigma_1}\frac{\lambda_2-\sigma_1}{\sigma_2-\sigma_1} = 0.2064 + 0.0013\mathrm{i}$$

$$p_2 = \frac{1}{\mathbf{b}^T\mathbf{v}_2}\frac{\lambda_2-\sigma_2}{\sigma_2}\frac{\lambda_1-\sigma_2}{\sigma_1-\sigma_2} = 0.2064 - 0.0013\mathrm{i}$$

Therefore, with $\mathbf{p} = [p_1 \quad p_2]^T$ Eqs. (5.17) and (5.18) yield

$$\mathbf{f} = \begin{pmatrix} -0.0011 \\ -0.0027 \\ -0.0085 \end{pmatrix} \text{ and } \mathbf{g} = \begin{pmatrix} -0.1213 \\ -0.1006 \\ -0.3763 \end{pmatrix}$$

The modified stiffness and damping matrices are obtained as;

$$\widetilde{\mathbf{K}} = \begin{bmatrix} 6.1213 & -1.8994 & -0.6237 \\ -2 & 4 & -2 \\ -0.8787 & -1.8994 & 3.3763 \end{bmatrix}$$

and

$$\widetilde{\mathbf{C}} = \begin{bmatrix} 0.1011 & 0.0027 & 0.0085 \\ 0 & 0.1 & -0.1 \\ 0.0011 & -0.0973 & 0.1085 \end{bmatrix}$$

The eigenvalues of the modified system are derived using the state-space form of the equations of motion.

$$\widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\widetilde{\mathbf{K}} & -\mathbf{M}^{-1}\widetilde{\mathbf{C}} \end{bmatrix}$$

Therefore,

$$\lambda_{1,2} = -0.005 \pm 0.7\mathrm{i}$$
$$\lambda_{3,4} = \sigma_{3,4} = -0.0374 \pm 1.6016\mathrm{i}$$
$$\lambda_{5,6} = \sigma_{5,6} = -0.0509 \pm 2.2638\mathrm{i}$$

The initial and modified receptances at m_1 are plotted in Figure 5.2, represented by the solid (blue) and dashed (red) line respectively.

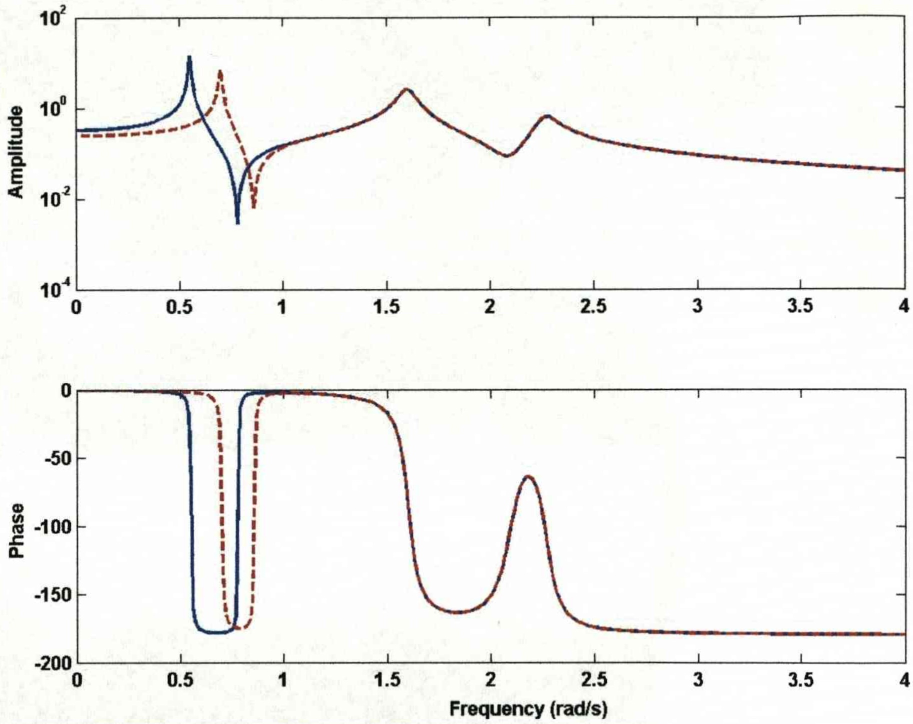


Figure 5.2. Example 5.1 - initial receptance (solid line) and modified receptance (dashed line)

5.3 Multiple-input case

In multi-input case the control force $\mathbf{u}(t)$ can be defined as;

$$\mathbf{u}(t) = -\mathbf{F}^T\dot{\mathbf{x}}(t) - \mathbf{G}^T\mathbf{x}(t) \quad (5.27)$$

where $\mathbf{F}, \mathbf{G} \in \Re^{n\times m}$. The solution for the case of multiple-input control is defined by Datta *et al.* [D6] as,

$$\mathbf{F} = -\mathbf{M}\mathbf{V}_1\mathbf{\Sigma}_1\mathbf{\Phi}^T \quad (5.28)$$

and

$$\mathbf{G} = \mathbf{K}\mathbf{V}_1\mathbf{\Phi}^T \quad (5.29)$$

where $\mathbf{\Sigma}_1 = \text{diag}(\sigma_1, \sigma_2, \ldots \sigma_p)$ and $\mathbf{V}_1 = \lfloor \mathbf{v}_1\, \mathbf{v}_2 \ldots \mathbf{v}_p \rfloor$ are the p eigenvalues and eigenvectors of the open-loop system to be changed. The assigned closed-loop eigenvalues are denoted by $\mathbf{\Lambda}_1 = \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_p)$.

Arbitrary vectors $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p$ are chosen in such a way that for the complex conjugate pairs of eigenvalues the associated $\boldsymbol{\gamma}_j$, $j = 1,2,\ldots,p$ are complex conjugate as well.

The equation $(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K})\mathbf{y}_j = \mathbf{B}\boldsymbol{\gamma}_j$, $j = 1,2,\ldots,p$ (5.29) is solved for $\mathbf{Y}_1 = \lfloor \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p \rfloor$ in order to form $\mathbf{Z}_1$ as in Eq. (5.30).

$$\mathbf{Z}_1 = \mathbf{\Lambda}_1\mathbf{Y}_1^T\mathbf{M}\mathbf{V}_1\mathbf{\Sigma}_1 - \mathbf{Y}_1^T\mathbf{K}\mathbf{V}_1 \quad (5.30)$$

and

$$\mathbf{\Phi}\mathbf{Z}_1^T = \mathbf{\Gamma} \quad (5.31)$$

where $\mathbf{\Phi} = \lfloor \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_p \rfloor$ and $\mathbf{\Gamma} = \lfloor \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p \rfloor$. (5.32)

The matrix $\mathbf{\Phi}$ is determined from Eq. (5.31) to determine control gains $\mathbf{F}$ and $\mathbf{G}$. It can be seen that the arbitrary choices of $\gamma_1, \gamma_2, \ldots, \gamma_p$ produces a family of solutions $\mathbf{Z}_1$. Therefore, the method gives the freedom in choosing the control gains in order to have a robust system. For a robust system the eigenvalues of the closed-loop system are less affected by the random perturbations of the feedback matrices. Kautsky and Nichols [K4] improved the robustness by increasing the condition number of $\mathbf{Z}_1$. The robustness of the closed-loop system is discussed in chapter eight.

Example 5.2.

The 3 dof system in example 1 is considered. Two actuators applies the force at m_1 and m_3. We wish to change the eigenvalues $\sigma_{1,2}$ to the new values $\lambda_{1,2} = -0.0045 \pm 0.75\text{i}$, while leaving the other four eigenvalues unchanged.

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

The eigenvalues of the open-loop system that need to be changed are;

$$\mathbf{\Sigma}_1 = \begin{bmatrix} -0.0033 + 0.5513\text{i} & 0 \\ 0 & -0.0033 - 0.5513\text{i} \end{bmatrix}$$

and the associated eigenvectors are;

$$\mathbf{V}_1 = \begin{bmatrix} 0.4831 - 0.0036\text{i} & 0.4831 + 0.0036\text{i} \\ 0.8026 + 0.0018\text{i} & 0.8026 - 0.0018\text{i} \\ 1 & 1 \end{bmatrix}$$

The eigenvalues $\mathbf{\Sigma}_1$ will be shifted to $\mathbf{\Lambda}_1$ as;

$$\mathbf{\Lambda}_1 = \begin{bmatrix} -0.0045+0.75\mathrm{i} & 0 \\ 0 & -0.0045-0.75\mathrm{i} \end{bmatrix}$$

The vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ are derived by choosing an arbitrary vector $\mathbf{\Gamma} = [\gamma_1 \quad \gamma_2]$ such that $\mathbf{\Gamma} = \begin{bmatrix} 1-2\mathrm{i} & 1+2\mathrm{i} \\ 1 & 1 \end{bmatrix}$.

Solving the equations;

$$\mathbf{y}_1 = \left(\lambda_1^2\mathbf{M} + \lambda_1\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{B}\gamma_1$$
$$\mathbf{y}_2 = \left(\lambda_2^2\mathbf{M} + \lambda_2\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{B}\gamma_2$$

yields;

$$\mathbf{Y}_1 = \begin{bmatrix} -0.5536+0.0845\mathrm{i} & -0.5536-0.0845\mathrm{i} \\ -1.1494+0.6585\mathrm{i} & -1.1494-0.6585\mathrm{i} \\ -1.4048+1.0609\mathrm{i} & -1.4048-1.0609\mathrm{i} \end{bmatrix}$$

The matrix $\mathbf{Z}_1$ and $\mathbf{\Phi}$ are obtained as;

$$\mathbf{Z}_1 = \begin{bmatrix} 4.0953-2.6910\mathrm{i} & -0.6286+0.4122\mathrm{i} \\ -0.6286-0.4122\mathrm{i} & 4.0953+2.6910\mathrm{i} \end{bmatrix}$$

and

$$\mathbf{\Phi} = \begin{bmatrix} 0.4661-0.1985\mathrm{i} & 0.4661+0.1985\mathrm{i} \\ 0.2015+0.0972\mathrm{i} & 0.2015-0.0972\mathrm{i} \end{bmatrix}$$

The control feedback matrices are found to be;

$$\mathbf{F} = \begin{bmatrix} -0.2122 & 0.1032 \\ -0.1723 & 0.0875 \\ -0.6474 & 0.3255 \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} 0.2637 & 0.1232 \\ 0.2331 & 0.0955 \\ 0.8501 & 0.3674 \end{bmatrix}$$

The modified stiffness and damping matrices are calculated;

$$\widetilde{\mathbf{K}} = \begin{bmatrix} 6.2637 & -1.7669 & -0.1499 \\ -2 & 4 & -2 \\ -0.8768 & -1.9045 & 3.3674 \end{bmatrix}$$

$$\text{and } \widetilde{\mathbf{C}} = \begin{bmatrix} 0.1122 & -0.1723 & -0.6474 \\ 0 & 0.1 & -0.1 \\ 0.1032 & -0.0125 & 0.4255 \end{bmatrix}$$

which yields to the following closed-loop eigenvalues from the state-space analysis;

$$\lambda_{1,2} = -0.0045 \pm 0.75i$$
$$\lambda_{3,4} = \sigma_{3,4} = -0.0374 \pm 1.6016i$$
$$\lambda_{5,6} = \sigma_{5,6} = -0.0509 \pm 2.2638i$$

A similar approach, based on single-input method, is developed by Ram *et al.* [R5] for use with multiple-inputs. In this approach a multi-input partial pole assignment is considered as a sequence of single-input control. The problem is solved by moving gradually the poles from their initial locations to their final destination.

5.4 Conclusion

In this chapter the theory of partial pole assignment problem by linear state feedback control is considered. The assignment is carried out using the single-input and multiple-input control. One of the well-known orthogonality relations is used to derive an explicit solution for a control system modelled by a second order differential equation. It is shown that for the single-input case a unique solution for the control gains is obtained and for the multi-input case a family of solutions is derived; thus allowing for robust solutions to be selected.

The method described in this chapter works by using the **M**, **C**, **K** matrices, typically obtained from finite elements that may include modelling errors, assumptions and approximations. In addition, model reduction or truncation for large system matrices is generally required. A new approach, the receptance method, is described in the following chapters. This approach has the significant advantage of avoiding the use of **M**, **C**, **K** matrices completely.

Chapter 6

POLE PLACEMENT BY THE RECEPTANCE METHOD USING STATE FEEDBACK CONTROL

6.0 Introduction

In this chapter a new theory based on receptance method is addressed for the assignment of poles and zeros in active vibration control. The theory, which is well known in passive structural modification, is used for the first time in active vibration control by Ram and Mottershead [R6]. In this approach the receptance transfer function is determined by inverting the dynamic stiffness matrix, but in practice would be obtained from the measured receptance frequency response function $\mathbf{H}(i\omega)$, so that there would be no need for the evaluation of the system matrices **M**, **C**, **K** typically obtained from finite elements. The method leads to linear characteristic equations in the unknown gains, a consequence of the state feedback used, resulting in a rank-1 change to the dynamic stiffness matrix. The application of receptance method is demonstrated by means of a series of numerical examples.

6.1 Introductory theory

The problem of assigning eigenvalues and eigenvectors of a vibrating system by feedback control has been developed since the 1960s when Wonham [W3] showed that poles of a system could be assigned by state feedback if the system was controllable. Miminis *et al.* [M7] developed algorithms for the pole assignment problem. Kautsky *et al.* [K3] described numerical algorithms to define robust solutions for the state-feedback pole assignment problem. Saad [S1] developed an algorithm for selective alteration of the eigenvalues using the first order differential equations. Datta *et al.* [D6] developed a closed-form solution for the partial pole placement in which some chosen eigenvalues were relocated and all the other eigenvalues remained unchanged. In the previous chapter the partial pole assignment problem was discussed based on the quadratic pencil with system matrices **M**, **C**, **K**. There are some disadvantages in using the system matrices **M**, **C**, **K**, which are typically obtained from finite element models. Firstly, the many different forms of damping in structures cannot all be represented by the standard form of second order matrix differential equations. FE models normally neglect damping or assume Rayleigh (proportional) damping. In active control the model of damping is very important in the eigenvalue assignment problem. The lack of damping can lead to an inaccurate feedback controller and may cause the closed-loop eigenvalues move to the unstable region of the complex plane. Secondly, FE models used in design can be very large therefore computationally expensive and require model reduction, truncation or other approximations which can degrade the performance of the controller. Finally the controller should be insensitive to the ill-defined FE parameters such as joints and boundary conditions. These uncertainties in the system parameters may result in lack of robustness in the closed-loop system. Stobener and Gaul [S8] used the experimental modal data instead of the FE models to extract modal parameters such as eigenvalues and eigenvectors from the measured data. However in their analysis proportional damping was considered. Ram and Mottershead [R6] introduced the use of receptances in active vibration control for the first time. In their approach the receptance transfer function is determined by inverting the dynamic stiffness matrix, but in practice would be obtained from the measured frequency response function $\mathbf{H}(\mathrm{i}\omega)$. The significant advantage of the receptance method over all the previous methods is that the receptance method does not require evaluating the system matrices **M**, **C**, **K**. Since the method uses receptances rather than dynamic stiffness, the system equations are complete with just a small number of states. This means that there is no need for model reduction or the estimator to estimate the unmeasured states using an observer. The use of receptance method in the inverse problem leads to linear characteristic equations in the unknown control gains for the assignment of poles, zeros or the assignment of poles and zeros together. The dynamic stiffness of the closed-loop system is changed by the rank-1 matrix, which is a consequence of state feedback using a single input $\mathbf{u}(t)$. The application of receptance method is demonstrated by means of a series of numerical examples.

We begin with writing the dynamic equation of the system including the state feedback control.

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{b}\mathbf{u}(t)+\mathbf{p}(t) \tag{6.1}$$

$$\mathbf{u}(t)=-\mathbf{f}^T\dot{\mathbf{x}}(t)-\mathbf{g}^T\mathbf{x}(t) \tag{6.2}$$

where $\mathbf{M},\mathbf{C},\mathbf{K}\in\Re^{n\times n}$; $\mathbf{v}^T\mathbf{M}\mathbf{v}>0$, $\mathbf{v}^T\mathbf{C}\mathbf{v}\geq 0$, $\mathbf{v}^T\mathbf{K}\mathbf{v}\geq 0$ for arbitrary $\mathbf{v}\in\Re^{n\times 1}$; and $\mathbf{b},\mathbf{p},\mathbf{f},\mathbf{g}\in\Re^{n\times 1}$. It should be noted that in practice each non-zero term in $\mathbf{b}$ implies the use of an actuator and each non-zero term in $\mathbf{g}$ or $\mathbf{f}$ implies the use of a sensor.

Combining equations (6.1) and (6.2) and taking Laplace transform yields;

$$\left(\mathbf{M}s^2+\mathbf{C}s+\mathbf{K}+\mathbf{b}(\mathbf{g}+s\mathbf{f})^T\right)\mathbf{x}(s)=\mathbf{p}(s) \tag{6.3}$$

It can be seen that the closed-loop dynamic stiffness is changed by the rank-1 matrix $\mathbf{b}(\mathbf{g}+s\mathbf{f})^T$, which is a consequence of state feedback using a single input $\mathbf{u}(t)$. Application of Sherman-Morrison formula [G7] gives the inverse of a matrix with a rank-1 modification in terms of the inverse of the original matrix. Hence, the closed-loop receptance matrix can be defined as,

$$\hat{\mathbf{H}}(s)=\mathbf{H}(s)-\frac{\mathbf{H}(s)\mathbf{b}(\mathbf{g}+s\mathbf{f})^T\mathbf{H}(s)}{1+(\mathbf{g}+s\mathbf{f})^T\mathbf{H}(s)\mathbf{b}} \tag{6.4}$$

where $\mathbf{H}(s)=[\mathbf{M}s^2+\mathbf{C}s+\mathbf{K}]^{-1}$ in practice is obtained from the matrix of measured receptances $\mathbf{H}(\mathrm{i}\omega)$ at the sensor/actuator coordinates typically by fitting a rational fraction polynomial to the measured $\mathbf{H}(\mathrm{i}\omega)$. The receptance terms to be measured occupy the submatrix corresponding to the non-zero terms in $\mathbf{b}$ and $\mathbf{g}+s\mathbf{f}$.

6.2 The pole assignment problem

The assignment of poles has potential applications in structural dynamics. Large-amplitude of vibrations close to resonances in structures can be avoided by moving the poles of the system to desired locations elsewhere. The poles of the system are the roots of the characteristic polynomial $1+(\mathbf{g}+s\mathbf{f})^T\mathbf{H}(s)\mathbf{b}$ in equation (6.4). In the case of assigning the full set of $2n$-poles to prescribed values $\{\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_{2n}\}$, the problem of pole assignment can be expressed as follows:

Given: $\mathbf{H}(s)$, $\mathbf{b}$, and a complex set $\{\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_{2n}\}$ closed under conjugation.

Find: $\mathbf{g}\in\Re^n$, $\mathbf{f}\in\Re^n$ such that $(\mathbf{g}+\lambda_k\mathbf{f})^T\mathbf{H}(\lambda_k)\mathbf{b}=-1$ for $k=1,\ldots,2n$.

Denoting,

$$\mathbf{r}_k=\mathbf{H}(\lambda_k)\mathbf{b} \tag{6.5}$$

Then we need to solve

$$\mathbf{r}_k^T(\mathbf{g}+\lambda_k\mathbf{f})=-1,\ k=1,...,2n \tag{6.6}$$

or

$$\mathbf{r}_k^T\mathbf{g}+\lambda_k\mathbf{r}_k^T\mathbf{f}=-1,\ k=1,...,2n \tag{6.7}$$

The set of $2n$ equations with $2n$ unknowns may be written in the matrix form,

$$\begin{bmatrix} \mathbf{r}_1^T & \lambda_1\mathbf{r}_1^T \\ \mathbf{r}_2^T & \lambda_2\mathbf{r}_2^T \\ \vdots & \vdots \\ \mathbf{r}_{2n}^T & \lambda_{2n}\mathbf{r}_{2n}^T \end{bmatrix}\begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix}=\begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \tag{6.8}$$

which allows determination of the control gains $\mathbf{g}$ and $\mathbf{f}$ by inverting the matrix $\mathbf{G}$.

$$\mathbf{G}=\begin{bmatrix} \mathbf{r}_1^T & \lambda_1\mathbf{r}_1^T \\ \mathbf{r}_2^T & \lambda_2\mathbf{r}_2^T \\ \vdots & \vdots \\ \mathbf{r}_{2n}^T & \lambda_{2n}\mathbf{r}_{2n}^T \end{bmatrix} \tag{6.9}$$

Control gains $\mathbf{g}$ and $\mathbf{f}$ define the control force $u(t)$ to be delivered via the distribution vector $\mathbf{b}$. In practice, actuators can be chosen appropriately in order to supply the force $\mathbf{b}u(t)$ in real time using the measured system response $\mathbf{x}(t), \dot{\mathbf{x}}(t)$.

The following theorems concern the solution of equation (6.8) and their proofs are given in [R6].

Theorem 1: $\mathbf{G}$ is invertible if the system is controllable and λ_k, $k=1,...,2n$, are distinct.

Theorem 2: If $\mathbf{G}$ is invertible and the set $\{\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_{2n}\}$ is closed under conjugation then $\mathbf{g}$ and $\mathbf{f}$ are real.

Proof of these theorems are given by Ram and Mottershead [R6].

Example 6.1

We wish to assign the poles of the system shown in Figure 6.1 with $\mathbf{b} = (1 \quad 1)^T$ to:

$$\lambda_{1,2} = -0.05 \pm 1.2\mathrm{i}$$
$$\lambda_{3,4} = -0.2 \pm 2.3\mathrm{i}$$

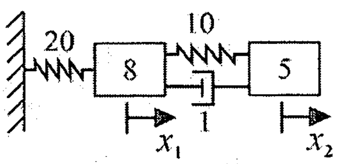


Figure 6.1. Mass-spring-damper system

In this case,

$$\mathbf{M} = \begin{bmatrix} 8 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 30 & -10 \\ -10 & 10 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The open-loop poles of the system are found to be;

$$\sigma_{1,2} = -0.0211 \pm 1.0349\mathrm{i}$$
$$\sigma_{3,4} = -0.1414 \pm 2.1556\mathrm{i}$$

We need to solve the characteristic equation (6.7). Therefore,

$$\mathbf{r}_1 = \left((-0.05+1.2\mathrm{i})^2\mathbf{M} + (-0.05+1.2\mathrm{i})\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{b} = \begin{pmatrix} -0.2642+0.0301\mathrm{i} \\ -0.5767+0.1191\mathrm{i} \end{pmatrix}$$
$$\mathbf{r}_2 = \left((-0.05-1.2\mathrm{i})^2\mathbf{M} + (-0.05-1.2\mathrm{i})\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{b} = \begin{pmatrix} -0.2642-0.0301\mathrm{i} \\ -0.5767-0.1191\mathrm{i} \end{pmatrix}$$
$$\mathbf{r}_3 = \left((-0.2+2.3\mathrm{i})^2\mathbf{M} + (-0.2+2.3\mathrm{i})\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{b} = \begin{pmatrix} -0.0465+0.0313\mathrm{i} \\ -0.0298-0.0080\mathrm{i} \end{pmatrix}$$
$$\mathbf{r}_4 = \left((-0.2-2.3\mathrm{i})^2\mathbf{M} + (-0.2-2.3\mathrm{i})\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{b} = \begin{pmatrix} -0.0465-0.0313\mathrm{i} \\ -0.0298+0.0080\mathrm{i} \end{pmatrix}$$

so that the matrix in equation (6.8) becomes,

$$\mathbf{G} = \begin{bmatrix} -0.2642+0.0301\mathrm{i} & -0.5767+0.1191\mathrm{i} & -0.0230-0.3185\mathrm{i} & -0.1140-0.6980\mathrm{i} \\ -0.2642-0.0301\mathrm{i} & -0.5767-0.1191\mathrm{i} & -0.0230+0.3185\mathrm{i} & -0.1140+0.6980\mathrm{i} \\ -0.0465+0.0313\mathrm{i} & -0.0298-0.0080\mathrm{i} & -0.0628-0.1133\mathrm{i} & 0.0244-0.0669\mathrm{i} \\ -0.0465-0.0313\mathrm{i} & -0.0298+0.0080\mathrm{i} & -0.0628+0.1133\mathrm{i} & 0.0244+0.0669\mathrm{i} \end{bmatrix}$$

and the solution then gives,

$$\mathbf{g} = \begin{pmatrix} 14.6480 \\ -4.6355 \end{pmatrix}$$

$$\mathbf{f} = \begin{pmatrix} 6.1250 \\ -2.9531 \end{pmatrix}$$

In order to validate the results, the eigenvalues of the closed loop state-space system, $\mathbf{A} - \lambda\mathbf{B}$ are found to be;

$$\lambda_{1,2} = -0.05 \pm 1.2\mathrm{i}$$
$$\lambda_{3,4} = -0.2 \pm 2.3\mathrm{i}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\left(\mathbf{K} + \mathbf{b}\mathbf{g}^T\right) & -\left(\mathbf{C} + \mathbf{b}\mathbf{f}^T\right) \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}$$

and the modified stiffness and damping matrices are:

$$\hat{\mathbf{K}} = \begin{bmatrix} 44.648 & -14 \\ 4.648 & 5.3645 \end{bmatrix}, \quad \hat{\mathbf{C}} = \begin{bmatrix} 7.125 & -3.9531 \\ 5.1250 & -1.9531 \end{bmatrix}$$

6.3 The zero assignment problem

The problem of zero assignment can be considered when the vibration response at chosen coordinates and frequencies should be vanished. The zeros for each receptance term are the roots of the characteristic polynomial of receptance $\hat{H}_{ij}$ obtained from the ij^{th} numerator term of equation (6.4).

Given: $\mathbf{H}(s)$, $\mathbf{b}$, i, j, and a complex set $\{\mu_1 \quad \mu_2 \quad \cdots \quad \mu_r\}$.

Find: $\mathbf{g}$, $\mathbf{f}$ such that

$$\mathbf{e}_i^T \left(\mathbf{H}(\mu_k) - \frac{\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g} + \mu_k \mathbf{f})^T \mathbf{H}(\mu_k)}{1 + (\mathbf{g} + \mu_k \mathbf{f})^T \mathbf{H}(\mu_k)\mathbf{b}} \right) \mathbf{e}_j = 0 \tag{6.10}$$

for $k = 1, \ldots, r$, where $\mathbf{e}_k$ is the unit vector formed from the k^{th} column of the identity matrix and $r \leq 2(n-1)$.

Solution:

The i,j^{th} term of the closed-loop transfer function matrix is given from equation (6.4) as,

$$\hat{H}_{ij}(s) = \frac{\mathbf{e}_i^T \left[\left(1 + (\mathbf{g} + s\mathbf{f})^T \mathbf{H}(s)\mathbf{b}\right)\mathbf{H}(s) - \mathbf{H}(s)\mathbf{b}(\mathbf{g} + s\mathbf{f})^T \mathbf{H}(s) \right] \mathbf{e}_j}{1 + (\mathbf{g} + s\mathbf{f})^T \mathbf{H}(s)\mathbf{b}} \tag{6.11}$$

We wish to find $\mathbf{f}$ and $\mathbf{g}$ such that,

$$\mathbf{e}_i^T\Big(\big(1+(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b}\big)\mathbf{H}(\mu_k)-\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\Big)\mathbf{e}_j=0 \tag{6.12}$$

for $k=1,2,\ldots,r$.

Equation (6.12) may be written in the form,

$$\begin{aligned}\big((\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b}\big)\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{e}_j-\mathbf{e}_i^T\big(\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\big)\mathbf{e}_j \\ =-\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{e}_j\end{aligned} \tag{6.13}$$

or,

$$\begin{aligned}\mathbf{H}_{ij}(\mu_k)(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b}-\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\,\mathbf{e}_j \\ =-\mathbf{H}_{ij}(\mu_k)\end{aligned} \tag{6.14}$$

with the obvious definition of $H_{ij}(s)$. Noting that $\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{b}$ is a scalar, we define

$$\mathbf{t}_k=\mathbf{H}_{ij}(\mu_k)\mathbf{H}(\mu_k)\mathbf{b}-\big(\mathbf{e}_i^T\mathbf{H}(\mu_k)\mathbf{b}\big)\mathbf{H}(\mu_k)\mathbf{e}_j \tag{6.15}$$

and write the linear system (6.15) for $k=1,\ldots,r$ in matrix form,

$$\begin{bmatrix}\mathbf{t}_1^T & \mu_1\mathbf{t}_1^T \\ \mathbf{t}_2^T & \mu_2\mathbf{t}_2^T \\ \vdots & \vdots \\ \mathbf{t}_r^T & \mu_r\mathbf{t}_r^T\end{bmatrix}\begin{pmatrix}\mathbf{g}\\ \mathbf{f}\end{pmatrix}=\begin{pmatrix}-H_{ij}(\mu_1)\\ -H_{ij}(\mu_2)\\ \vdots \\ -H_{ij}(\mu_r)\end{pmatrix} \tag{6.16}$$

which allows determination of $\mathbf{g}$ and $\mathbf{f}$ as functions depending on $2n-r$ arbitrary constants. Of course, only real $\mathbf{g}$ and $\mathbf{f}$ are realizable control parameters.

Theorem 3: If $i = j$ and the set $\{\mu_1 \quad \mu_2 \quad \cdots \quad \mu_r\}$ is closed under conjugation then $\mathbf{g}$ and $\mathbf{f}$ can be chosen as real. The proof is given by Ram and Mottershead [R6].

Example 6.2.

With $\mathbf{b} = (1 \quad 1)^T$, we wish to assign the zeros of $\mathbf{H}_{11}(s)$ of the system shown in Figure 6.1 to: $\mu_1 = -0.1 + 2\mathrm{i}$, $\mu_2 = -0.1 - 2\mathrm{i}$.

Solution:

Using equation (6.15) we obtain,

$$\mathbf{t}_1 = \begin{pmatrix} 0 \\ -0.0051 + 0.005\mathrm{i} \end{pmatrix}$$

$$\mathbf{t}_2 = \begin{pmatrix} 0 \\ -0.0051 - 0.005\mathrm{i} \end{pmatrix}$$

so that by equation (6.16),

$$\mathbf{G} = \begin{bmatrix} 0 & -0.0051 + 0.005\mathrm{i} & 0 & -0.0104 - 0.0096\mathrm{i} \\ 0 & -0.0051 - 0.005\mathrm{i} & 0 & -0.0104 + 0.0096\mathrm{i} \end{bmatrix}$$

The solution of equation (6.16) is obtained using pseudo-inverse,

$$\mathbf{g} = \begin{pmatrix} 0 \\ 10.05 \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

In order to validate the results, the transfer functions of the closed loop system are determined;

$$\mathbf{H}(-0.1+2\mathrm{i}) = \left((-0.1+2\mathrm{i})^2\mathbf{M} + (-0.1+2\mathrm{i})(\mathbf{C}+\mathbf{b}\mathbf{f}^T) + \mathbf{K} + \mathbf{b}\mathbf{g}^T\right)^{-1}$$
$$= \begin{bmatrix} 0 & -0.0953+0.0189\mathrm{i} \\ -0.0069+0.1664\mathrm{i} & -0.0876-0.0162\mathrm{i} \end{bmatrix}$$

and

$$\mathbf{H}(-0.1-2\mathrm{i}) = \left((-0.1-2\mathrm{i})^2\mathbf{M} + (-0.1-2\mathrm{i})(\mathbf{C}+\mathbf{b}\mathbf{f}^T) + \mathbf{K} + \mathbf{b}\mathbf{g}^T\right)^{-1}$$
$$= \begin{bmatrix} 0 & -0.0953-0.0189\mathrm{i} \\ -0.0069-0.1664\mathrm{i} & -0.0876+0.0162\mathrm{i} \end{bmatrix}$$

and it can be seen that $h_{11}(-0.1+2\mathrm{i}) = h_{11}(-0.1-2\mathrm{i}) = 0$ as required.

6.4 Assignment of poles and zeros

The pole and zero assignment can also be assigned together. One application of simultaneous assignment of poles and zeros is the assignment of vibration nodes. When a pole and a zero coincide at the same eigenvalue on the imaginary axis the pole-zero cancellation occurs, thus creating a vibration node.

Given: $\mathbf{H}(s)$, $\mathbf{b}$, i, j, and two complex set $\{\mu_1 \quad \mu_2 \quad \cdots \quad \mu_r\}$ and $\{\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_p\}$.

Find: $\mathbf{g}$, $\mathbf{f}$ such that

$$\mathbf{e}_i^T\left(\mathbf{H}(\mu_k) - \frac{\mathbf{H}(\mu_k)\mathbf{b}(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)}{1+(\mathbf{g}+\mu_k\mathbf{f})^T\mathbf{H}(\mu_k)\mathbf{b}}\right)\mathbf{e}_j = 0,\ k = 1,\ldots,r \qquad (6.17)$$

$$(\mathbf{g}+\lambda_k\mathbf{f})^T\mathbf{H}(\lambda_k)\mathbf{b} = -1,\ \ k = 1,\ldots,p \qquad (6.18)$$

where $r + p \le 2n$.

Solution:

We need to solve,

$$\begin{bmatrix} \mathbf{t}_1^T & \mu_1\mathbf{t}_1^T \\ \mathbf{t}_2^T & \mu_2\mathbf{t}_2^T \\ \vdots & \vdots \\ \mathbf{t}_r^T & \mu_r\mathbf{t}_r^T \\ \mathbf{r}_1^T & \lambda_1\mathbf{r}_1^T \\ \mathbf{r}_2^T & \lambda_2\mathbf{r}_2^T \\ \vdots & \vdots \\ \mathbf{r}_p^T & \lambda_p\mathbf{r}_p^T \end{bmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} -H_{ij}(\mu_1) \\ -H_{ij}(\mu_2) \\ \vdots \\ -H_{ij}(\mu_r) \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \tag{6.19}$$

where $\mathbf{t}_k$, $k = 1,\ldots,r$ are given by equation (6.15) and,

$$\mathbf{r}_k = \mathbf{H}(\lambda_k)\mathbf{b}, \quad k = 1,\ldots,p \tag{6.20}$$

Example 6.3.

With $\mathbf{b} = (1 \quad 1)^T$, we wish to assign the poles $\lambda_{1,2} = \pm 1.5\mathrm{i}$, and the zeros of $h_{11}(s)$ to $\mu_{1,2} = \pm 1.5\mathrm{i}$, thereby eliminating the pole at 6 rad/sec with a zero at the same frequency.

Solution:

From equations (6.17) and (6.18) it is found that,

$$\mathbf{G} = \begin{bmatrix} 0 & -0.0086+0.001\mathrm{i} & 0 & -0.0016-0.0129\mathrm{i} \\ 0 & -0.0086-0.001\mathrm{i} & 0 & -0.0016+0.0129\mathrm{i} \\ -0.0781-0.0167\mathrm{i} & -0.1917-0.003\mathrm{i} & -0.025-0.1171\mathrm{i} & 0.0044-0.2875\mathrm{i} \\ -0.0781+0.0167\mathrm{i} & -0.1917+0.003\mathrm{i} & -0.025+0.1171\mathrm{i} & 0.0044+0.2875\mathrm{i} \end{bmatrix}$$

which yields the solution,

$$\mathbf{g} = \begin{pmatrix} 10 \\ 1.25 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The other poles of the system are $\lambda_{3,4} = -0.125 \pm 2.2326\mathrm{i}$. The initial receptance and modified receptance h_{11} are plotted in Figure 6.2. The pole-zero cancellation is occurred on the imaginary axis resulting in a vibration node at the first mass. A small blip at 1.5 rad/sec is due to digital calculation with a finite frequency resolution.

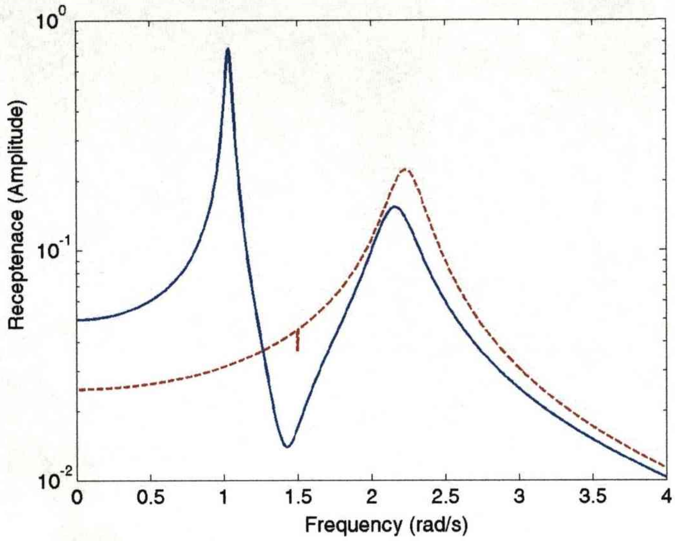


Figure 6.2. Original (solid line) and assigned (dashed line) receptance h_{11}

6.5 Conclusion

In this chapter the receptance method in active vibration control using state feedback is discussed. The receptance matrix is determined by inverting the dynamic stiffness, however in practice, is obtained from the measured frequency response function $\mathbf{H}(\mathrm{i}\omega)$. Therefore, there is no need for the evaluation of the system matrices **M**, **C**, **K** that are normally obtained from finite element models. The characteristic equations, which are linear in the unknown system gains, are derived for the assignment of poles, zeros and poles and zeros together. The closed-loop dynamic stiffness is changed by the rank-1 modification as a consequence of the single input state feedback control.

Chapter 7

POLE PLACEMENT BY THE RECEPTANCE METHOD USING OUTPUT FEEDBACK CONTROL

7.0 Introduction

In this chapter the receptance method is applied to output-feedback control [M16]. Measured receptances are used in place of the $\mathbf{M}, \mathbf{C}, \mathbf{K}$ matrices, usually obtained from finite elements, which may be large, require reduction, and include very considerable uncertainty due to modelling assumptions. Ram and Mottershead [R6] showed how state-feedback control using measured receptances from the original (open-loop) system could be used to assign all the poles of the closed loop system using just a single actuator. Poles (resonances) and zeros (antiresonances) were assigned without the need for the $\mathbf{M}, \mathbf{C}, \mathbf{K}$ matrices. The very considerable advantage of the output feedback method over state feedback is that it allows the use of collocated actuators and sensors in multiple-input-multiple-output (MIMO) systems. It is well known that the transfer function of a lightly damped system with a collocated actuator-sensor pair displays a line of interlacing poles and zeros just to the left of the imaginary axis. In addition to velocity feedback, for active damping, the method uses displacement feedback, for active stiffness, thereby enabling the assignment of both poles and zeros to desired locations in the complex s-plane. The assignment of zeros is of special interest in vibration analysis because the vibration can be made to vanish at chosen frequencies and locations. Application of the method is demonstrated by numerical examples.

7.1 Poles and zeros

The dynamical equations, in the Laplace frequency domain, may be cast in form of the second order matrix equation,

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{x}(s) = \mathbf{B}\mathbf{u}(s) + \mathbf{p}(s) \tag{7.1}$$

where, $\mathbf{K}, \mathbf{C}, \mathbf{M} \in \Re^{n \times n}$ are the usual structural stiffness, damping and mass matrices, $\mathbf{B} \in \Re^{n \times m}$ is the control force distribution matrix, $\mathbf{x}(s), \mathbf{p}(s) \in \Re^{n \times 1}$ represent the displacement states and external forces respectively and $\mathbf{u}(s) \in \Re^{m \times 1}$ is the control force. Likewise, the output equation may be written as,

$$\mathbf{y}(s) = \mathbf{D}\mathbf{x}(s) \tag{7.2}$$

where $\mathbf{D} \in \Re^{l \times n}$ is the sensor distribution matrix and the output is denoted by $\mathbf{y} \in \Re^{l \times 1}$. The feedback law is expressed as,

$$\mathbf{u}(s) = -(\mathbf{G} + s\mathbf{F})\mathbf{y}(s) \tag{7.3}$$

so that the output and rate gains are given by the terms in the matrices $\mathbf{G}, \mathbf{F} \in \Re^{m \times l}$.

A constraint is now applied in the form of,

$$\mathbf{D} = \mathbf{B}^T \in \Re^{m \times n} \tag{7.4}$$

which defines the most general condition of collocated actuators and sensors. This restriction has considerable practical advantages for the avoidance instability as explained by Preumont [P5]. The resulting system of equations retains considerably greater freedom in the choice of control gains than does a passive structural modification by the adjustment of physical parameters such as beam cross sections or added masses.

Combining equations (7.1-7.4) leads to,

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} + \mathbf{B}(\mathbf{G} + s\mathbf{F})\mathbf{B}^T\right)\mathbf{x}(s) = \mathbf{p}(s) \tag{7.5}$$

In principle if $\left(\mathbf{K} + \mathbf{BGB}^T\right)$, $\left(\mathbf{C} + \mathbf{BFB}^T\right) \geq \mathbf{0}$ and $\mathbf{M} > 0$, then the real part of the closed-loop system eigenvalues will be strictly negative.

A particular case is to choose the control gain matrices to be diagonal, thereby reducing to a minimum the number of gain terms to be found,

$$\mathbf{F} = \mathrm{diag}(f_i); \quad \mathbf{G} = \mathrm{diag}(g_i); \quad i = 1,2,...m \tag{7.6}$$

Then, using equation (7.6) leads to,

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} + \mathbf{B}\mathrm{diag}(g_i + sf_i)\mathbf{B}^T\right)\mathbf{x}(s) = \mathbf{p}(s) \tag{7.7}$$

from which it is seen that the dynamic stiffness matrix $\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} + \mathbf{B}\mathrm{diag}(g_i + sf_i)\mathbf{B}^T\right)$ remains symmetric for all values of the gains g_i, f_i and the interlacing property of the poles and point-receptance zeros is maintained.

Our purpose is to assign poles λ_j of the closed-loop system determined by the solution of,

$$\det\left(\lambda_j^2\mathbf{M} + \lambda_j\left(\mathbf{C} + \mathbf{B}\mathrm{diag}(f_i)\mathbf{B}^T\right) + \left(\mathbf{K} + \mathbf{B}\mathrm{diag}(g_i)\mathbf{B}^T\right)\right) = 0, \quad j = 1,2,\ldots,r; \quad r \leq 2\mathrm{n} \tag{7.8}$$

which we assume to be distinct and closed under conjugation. A necessary condition is that the matrices $\mathbf{M}$, $\left(\mathbf{C} + \mathbf{B}\mathrm{diag}(f_i)\mathbf{B}^T\right)$ and $\left(\mathbf{K} + \mathbf{B}\mathrm{diag}(g_i)\mathbf{B}^T\right)$ should all be real and symmetric.

The closed-loop zeros, μ_k of the pp^{th} point receptance, denoted $h_{pp}(s)$, are given by the solution of a different eigenvalue problem,

$$\det\left(\mu_k^2\mathbf{M}_p + \mu_k\left(\mathbf{C} + \mathbf{B}\operatorname{diag}(f_i)\mathbf{B}^T\right)_p + \left(\mathbf{K} + \mathbf{B}\operatorname{diag}(g_i)\mathbf{B}^T\right)_p\right) = 0 \qquad (7.9)$$

where $\mathbf{M}_p$ is the submatrix corresponding to the pp^{th} term of $\mathbf{M}$. Thus,

$$\mathbf{M} = \begin{bmatrix} m_{pp} & \mathbf{m}_p^T \\ \mathbf{m}_p & \mathbf{M}_p \end{bmatrix}; \quad \mathbf{m}_p^T = \begin{bmatrix} m_{p1} & \cdots & m_{p,p-1} & m_{p,p+1} & \cdots & m_{pn} \end{bmatrix}$$

(7.10), (7.11)

$$\text{and } \mathbf{M}_p = \begin{bmatrix} m_{11} & \cdots & m_{1,p-1} & m_{1,p+1} & \cdots & m_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m_{p-1,1} & \cdots & m_{p-1,p-1} & m_{p-1,p+1} & \cdots & m_{p-1,n} \\ m_{p+1,1} & \cdots & m_{p+1,p-1} & m_{p+1,p+1} & \cdots & m_{p+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m_{n1} & \cdots & m_{n,p-1} & m_{n,p+1} & \cdots & m_{nn} \end{bmatrix} \qquad (7.12)$$

with similar definitions for the submatrices $\left(\mathbf{C} + \mathbf{B}\operatorname{diag}(f_i)\mathbf{B}^T\right)_p$ and $\left(\mathbf{K} + \mathbf{B}\operatorname{diag}(g_i)\mathbf{B}^T\right)_p$. It is seen that the zeros are the eigenvalues of the system grounded at the p^{th} coordinate. As with the poles, we assume the zeros to be distinct and closed under conjugation.

7.2 Pole and zero assignment by using receptances

The receptance matrix of the open-loop system is now defined as,

$$\mathbf{H}(s) = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1} \qquad (7.13)$$

Premultiplying both sides of equation (7.7) by $\mathbf{H}(s)$ then gives,

$$(\mathbf{I}+\mathbf{H}(s)\Delta\mathbf{Z}(s))\mathbf{x}(s)=\mathbf{H}(s)\mathbf{p}(s) \tag{7.14}$$

where, $\Delta\mathbf{Z}(s)=\mathbf{B}\mathrm{diag}(g_i+sf_i)\mathbf{B}^T$ (7.15)

and finally the closed-loop receptance equation can expressed in terms of the open-loop receptances as,

$$\mathbf{x}(s)=\left(\mathbf{I}+\mathbf{H}(s)\mathbf{B}\mathrm{diag}\left(g_i+sf_i\right)\mathbf{B}^T\right)^{-1}\mathbf{H}(s)\mathbf{p}(s) \tag{7.16}$$

or

$$\mathbf{x}(s)=\frac{\mathrm{adj}\left(\mathbf{I}+\mathbf{H}(s)\mathbf{B}\mathrm{diag}\left(g_i+sf_i\right)\mathbf{B}^T\right)}{\det\left(\mathbf{I}+\mathbf{H}(s)\mathbf{B}\mathrm{diag}\left(g_i+sf_i\right)\mathbf{B}^T\right)}\mathbf{H}(s)\mathbf{p}(s) \tag{7.17}$$

The closed-loop system poles may be assigned by selecting real-valued gains g_i, f_i, to satisfy the nonlinear characteristic equations,

$$\det\left(\mathbf{I}+\mathbf{H}(\lambda_j)\mathbf{B}\,\mathrm{diag}\left(g_i+\lambda_j f_i\right)\mathbf{B}^T\right)=0; \quad j=1,2,\ldots,r; \quad r\le 2n \tag{7.18}$$

where the assigned poles are distinct and closed under conjugation. Since the equations are nonlinear in the gains there may be one or more strictly real solutions $g_i, f_i, i=1,2,\ldots,m$ or there may be no solution, in which case the closed-loop poles may not be assigned to the prescribed values.

The zeros of the closed-loop system occur when terms in the numerator matrix product, $\mathrm{adj}\left(\mathbf{I}+\mathbf{H}(s)\mathbf{B}\mathrm{diag}\left(g_i+sf_i\right)\mathbf{B}^T\right)\mathbf{H}(s)$ vanish to zero. The zeros of the pp^{th} receptance may therefore be assigned by selecting gains such that,

$$\left[\mathrm{adj}\left(\mathbf{I}+\mathbf{H}(\mu_k)\mathbf{B}\mathrm{diag}\left(g_i+\mu_k f_i\right)\mathbf{B}^T\right)\mathbf{H}(\mu_k)\right]_{pp}=0 \tag{7.19}$$

We consider zeros that are distinct and closed under conjugation, so that if a solution (or solutions) exist then the gains are found to be strictly real.

7.3 Numerical examples

The method is illustrated by a series of eigenvalue assignment exercises based upon the system shown in Figure 7.1. The parameters of the system have the following values: $k_1 = 3, k_2 = 2,\ k_3 = 2,\ k_4 = 1,\ c_1 = 0.1,\ c_2 = 0.1,\ m_1 = 2, m_2 = 1,\ m_3 = 3.$

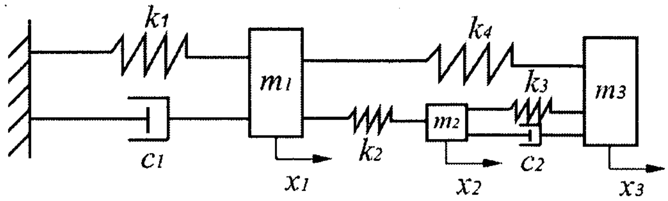


Figure 7.1. Three degree-of-freedom system

Example 7.1: Assignment of poles

Two pairs of complex conjugate poles at $\lambda_{1,2} = -0.01 \pm 0.7\mathrm{i}$ and $\lambda_{3,4} = -0.06 \pm 1.8\mathrm{i}$ are assigned using two actuators supplying feedback control forces at masses m_1 and m_3. The number of actuators is chosen to be less than the number of states in the system.

The mass, damping and stiffness matrices are given by;

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},\quad \mathbf{C} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & -0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix},\quad \mathbf{K} = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$

and the matrices $\mathbf{B}$ and $\mathbf{D}$ are chosen to be,

$$\mathbf{D} = \mathbf{B}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that $y_1 = x_1, y_2 = x_3$.

The open-loop system receptances were at the values of s corresponding to the chosen eigenvalues,

$$\mathbf{H}(\lambda_j) = \left(\mathbf{M}\lambda_j^2 + \mathbf{C}\lambda_j + \mathbf{K}\right)^{-1}; \quad j = 1,\ldots,4$$

Four characteristic equations (7.18), generally nonlinear in the gains, g_i, f_i, are then solved numerically using a Gauss-Newton method,

$$\det\left(\mathbf{I} + \mathbf{H}\left(\lambda_j\right)\mathbf{B}\,\mathrm{diag}\left(g_i + \lambda_j f_i\right)\mathbf{B}^T\right) = 0; \quad j = 1,\ \ldots, 4$$

with the result that,

$$\mathbf{F} = \mathrm{diag}\left(0.0999,\, 0.0475\right), \quad \mathbf{G} = \mathrm{diag}\left(2.3873,\, 0.3401\right)$$

In order to validate the results, the eigenvalues of the closed-loop system are obtained by the state space method.

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K} + \mathbf{B}\,\mathrm{diag}(g_i)\mathbf{B}^T) & -\mathbf{M}^{-1}(\mathbf{C} + \mathbf{B}\,\mathrm{diag}(f_i)\mathbf{B}^T) \end{bmatrix}$$

where

$$\left(\mathbf{C} + \mathbf{B}\,\mathrm{diag}(f_i)\mathbf{B}^T\right) = \begin{bmatrix} 0.1999 & 0 & 0 \\ 0 & 0.1 & -0.1 \\ 0 & -0.1 & 0.1475 \end{bmatrix}$$

and

$$\left(\mathbf{K} + \mathbf{B}\,\mathrm{diag}(g_i)\mathbf{B}^T\right) = \begin{bmatrix} 8.3873 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3.3401 \end{bmatrix}$$

which yields the following poles;

$$\lambda_{1,2} = -0.0100 \pm 0.7000\text{i}$$
$$\lambda_{3,4} = -0.0600 \pm 1.8000\text{i}$$
$$\lambda_{5,6} = -0.0546 \pm 2.3599\text{i}$$

The first two pairs exactly replicate the values assigned and the third pair of poles is stable. The initial and modified receptances at m_1 are plotted in Figure 7.2, represented by the solid (blue) and dashed (red) line respectively.

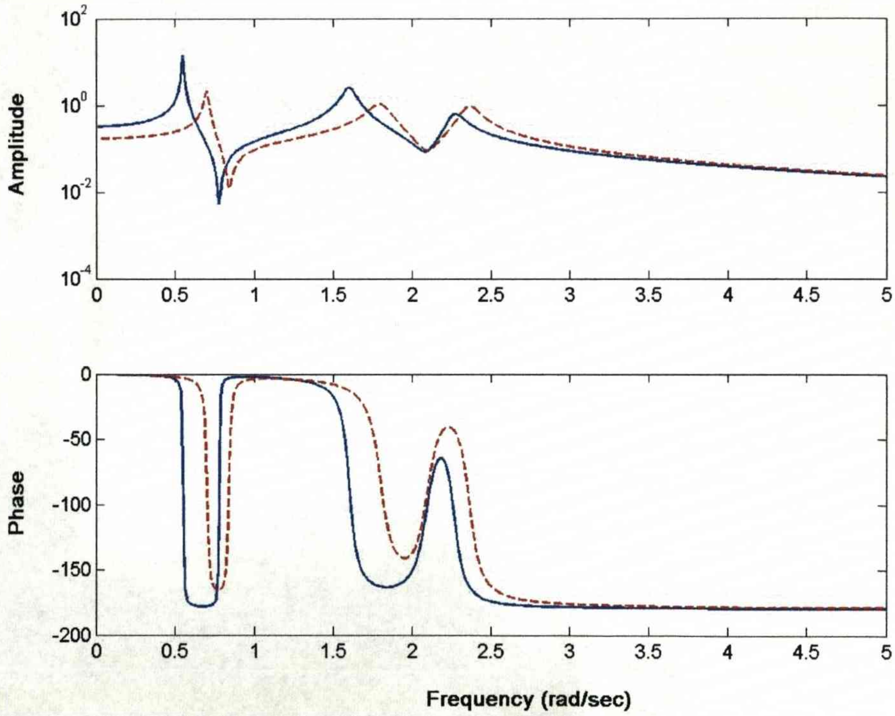


Figure 7.2. Initial receptance (solid line) and modified receptance (dashed line)

Example 7.2: Assignment of zeros

The zero assignment is considered for the point receptance h_{22}. Two pairs of complex conjugate zeros $\mu_{1,2} = -0.025 \pm 1.2\mathrm{i}$ and $\mu_{3,4} = -0.037 \pm 2\mathrm{i}$ are assigned using the feedback control forces at masses m_1 and m_3.

The sets of nonlinear equations (7.19) are solved to obtain the gains f_i and g_i.

$$\left[\mathrm{adj}\left(\mathbf{I} + \mathbf{H}(\mu_j)\mathbf{B}\,\mathrm{diag}\left(g_i + \mu_j f_i\right)\mathbf{B}^T\right)\mathbf{H}(\mu_j)\right]_{22} = 0; \quad j = 1,\cdots,4$$

which yields;

$$\mathbf{F} = \mathrm{diag}(0.0493, 0.048),\ \mathbf{G} = \mathrm{diag}(1.8691, 1.5223)$$

The zeros, being the eigenvalues of the system grounded at the 2nd coordinate, may be obtained by the state space method using the gains determined above. Thus,

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_2^{-1}\left(\mathbf{K} + \mathbf{B}\,\mathrm{diag}(g_i)\mathbf{B}^T\right)_2 & -\mathbf{M}_2^{-1}\left(\mathbf{C} + \mathbf{B}\,\mathrm{diag}(f_i)\mathbf{B}^T\right)_2 \end{bmatrix}$$

where

$$\left(\mathbf{C} + \mathbf{B}\,\mathrm{diag}(f_i)\mathbf{B}^T\right)_2 = \begin{bmatrix} 0.1493 & 0 \\ 0 & 0.1480 \end{bmatrix}$$

and

$$\left(\mathbf{K} + \mathbf{B}\,\mathrm{diag}(g_i)\mathbf{B}^T\right)_2 = \begin{bmatrix} 7.8691 & -1 \\ -1 & 4.5223 \end{bmatrix}$$

which yields the zeros,

$$\mu_{1,2} = -0.025 \pm 1.2i$$
$$\mu_{3,4} = -0.037 \pm 2i$$

for the point receptance h_{22}. The initial and modified receptances, plotted in Figure 7.3, are represented by the solid (blue) and dashed (red) lines respectively.

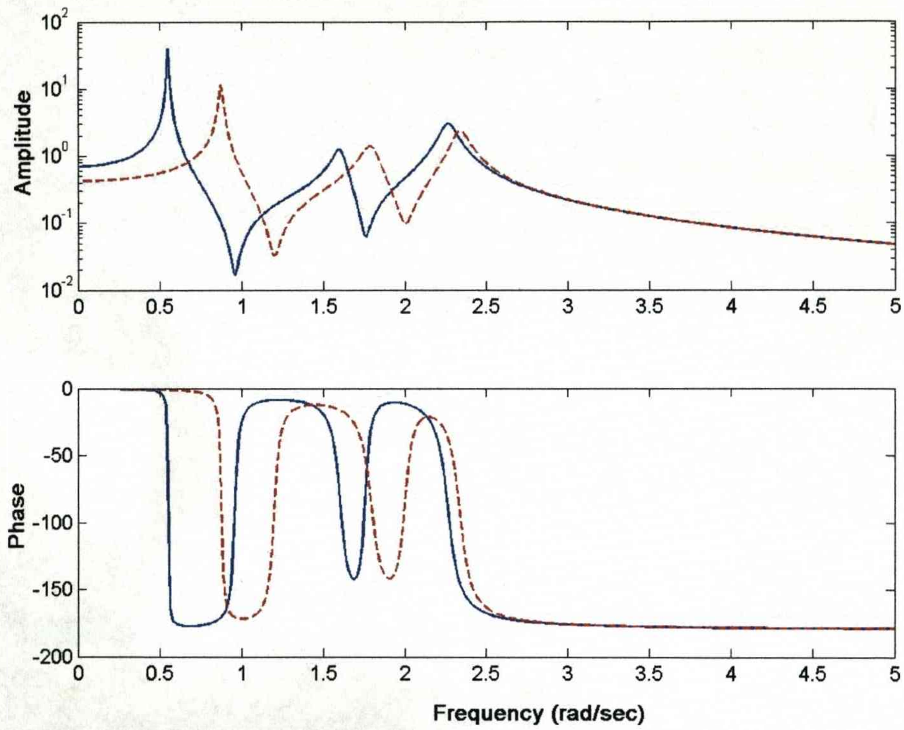


Figure 7.3. Initial receptance (solid line) and modified receptance (dashed line)

Example 7.3: Assignment of poles and zeros together

Poles and zeros are assigned at to h_{11} at $\lambda_{1,2} = -0.02 \pm 0.7\mathrm{i}$ and $\mu_{1,2} = -0.008 \pm 0.9\mathrm{i}$ respectively. Four nonlinear equations are solved simultaneously.

$$\det\left(\mathbf{I} + \mathbf{H}(\lambda_1)\mathbf{B}\operatorname{diag}(f_i + \lambda_1 g_i)\mathbf{B}^T\right) = 0$$

$$\det\left(\mathbf{I} + \mathbf{H}(\lambda_2)\mathbf{B}\operatorname{diag}(f_i + \lambda_2 g_i)\mathbf{B}^T\right) = 0$$

$$\left[\operatorname{adj}\left(\mathbf{I} + \mathbf{H}(\mu_1)\mathbf{B}\operatorname{diag}(g_i + \mu_1 f_i)\mathbf{B}^T\right)\mathbf{H}(\mu_1)\right]_{11} = 0$$

$$\left[\operatorname{adj}\left(\mathbf{I} + \mathbf{H}(\mu_2)\mathbf{B}\operatorname{diag}(g_i + \mu_2 f_i)\mathbf{B}^T\right)\mathbf{H}(\mu_2)\right]_{11} = 0$$

which yields to the following gains;

$$\mathbf{F} = \operatorname{diag}(0.4023, 0.0404),\ \mathbf{G} = \operatorname{diag}(0.3720, 0.6836)$$

State-space analysis, using these gains, results in the poles,

$$\begin{aligned}
\lambda_{1,2} &= -0.0200 \pm 0.7000\mathrm{i} \\
\lambda_{3,4} &= -0.0992 \pm 1.6497\mathrm{i} \\
\lambda_{5,6} &= -0.0798 \pm 2.2755\mathrm{i}
\end{aligned}$$

and zeros,

$$\begin{aligned}
\mu_{1,2} &= -0.0080 \pm 0.9000\mathrm{i} \\
\mu_{3,4} &= -0.0654 \pm 2.1007\mathrm{i}
\end{aligned}$$

The initial and modified receptances h_{11} are plotted in Figure 7.4, represented by the solid (blue) and dashed (red) line respectively.

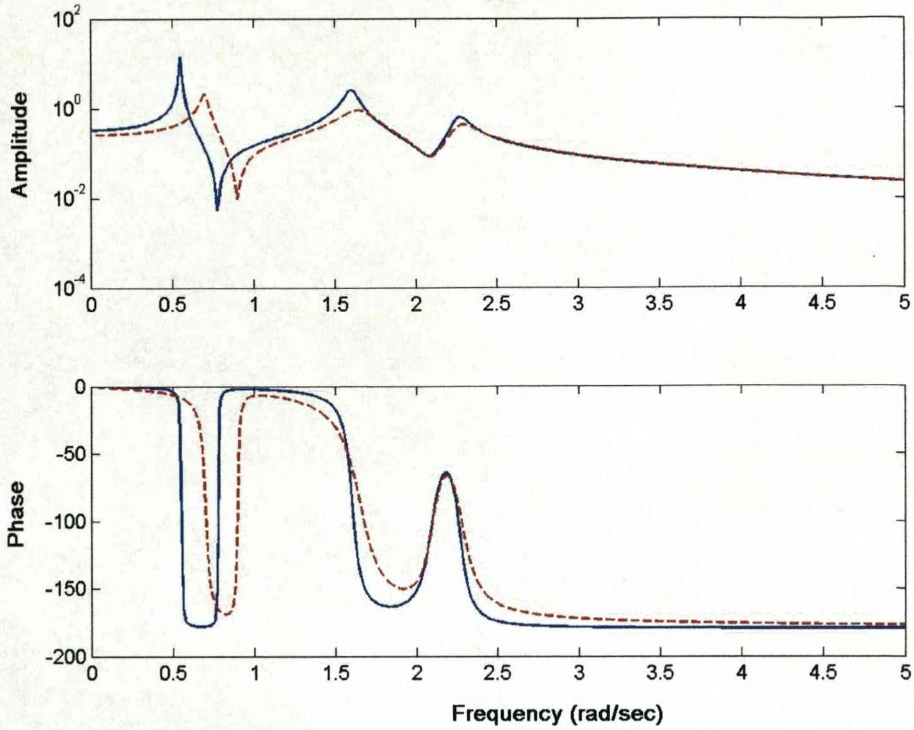


Figure 7.4. Initial receptance (solid line) and modified receptance (dashed line)

7.4 Fully populated gain matrices

In the previous numerical examples the gain matrices are diagonal,

$$\mathbf{F} = \operatorname{diag}(f_i); \quad \mathbf{G} = \operatorname{diag}(g_i); \quad i = 1,2,\ldots m$$

This is perhaps the simplest form for the gain matrices, but it should be noted that in more complex or larger structural problems there may be advantages in retaining the fully-populated symmetric form of feedback control gains. In principle, the fully-populated matrices should allow more poles and zeros to be assigned than the number of actuators and sensors. Alternatively, a greater number of solutions to the nonlinear characteristic equations may become available, thereby allowing the selection of gains that result in the least cost of control or that render those assigned poles and zeros least sensitive to small changes in the gain terms.

Example 7.4: Assignment of poles using fully populated control gain

Three pairs of complex conjugate poles using the fully populated matrices are assigned using the two actuators.

$$\lambda_{1,2} = -0.0100 \pm 0.7\mathrm{i}$$
$$\lambda_{3,4} = -0.0600 \pm 1.8\mathrm{i}$$
$$\lambda_{5,6} = -0.1 \pm 2.4\mathrm{i}$$

The control gains are found to be;

$$\mathbf{F} = \begin{bmatrix} 3.0657 & 0.2551 \\ 0.2551 & -0.0785 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 3.0667 & 0.5457 \\ 0.5757 & -0.0322 \end{bmatrix}$$

The plot of modified receptance in Figure 7.5 indicates the assignment of poles at their prescribed locations.

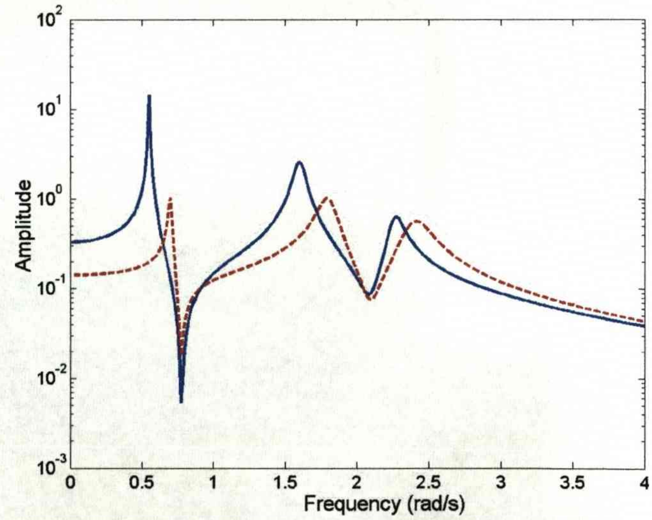


Figure 7.5. Initial receptance (solid line) and modified receptance (dashed line)

The modified stiffness and damping matrices are obtained as;

$$\widetilde{\mathbf{K}} = \begin{bmatrix} 9.0667 & -2 & -0.4543 \\ -2 & 4 & -2 \\ -0.4543 & -2 & 2.9678 \end{bmatrix}, \widetilde{\mathbf{C}} = \begin{bmatrix} 0.4657 & 0 & 0.2551 \\ 0 & 0.1 & -0.1 \\ 0.2551 & -0.1 & 0.0215 \end{bmatrix}$$

which leads to the assignment of the prescribed three pairs of complex conjugate poles.

Example 7.5: Assignment of poles using symmetric positive-definite fully populated control gain

If the symmetric fully populated control gains $\mathbf{F}$ and $\mathbf{G}$ are considered to be positive definite then the stability of the closed loop system is guaranteed.

If we assume that $\mathbf{F}$ is symmetric positive definite; then there exists exactly one lower triangular matrix $\mathbf{A}$ with the positive leading diagonal such that;

$$\mathbf{F} = \mathbf{A}\mathbf{A}^T \tag{7.20}$$

where,

$$\mathbf{F} = \begin{bmatrix} f_1 & f_3 \\ f_3 & f_2 \end{bmatrix} \tag{7.21}$$

Cholesky decomposition of $\mathbf{F}$ leads to;

$$\mathbf{A} = \begin{bmatrix} \sqrt{f_1} & 0 \\ \dfrac{f_3}{\sqrt{f_1}} & \sqrt{f_2 - \dfrac{f_3^2}{f_1}} \end{bmatrix} \tag{7.22}$$

The following constraints are defined from matrix $\mathbf{A}$ in order to have positive definite control gains;

$$\begin{aligned} & f_1 > 0 \\ & f_2 - \frac{f_3^2}{f_1} > 0 \end{aligned} \qquad (7.23), (7.24)$$

The same constraints are considered for the control gain matrix $\mathbf{G}$.

$$\begin{aligned} & g_1 > 0 \\ & g_2 - \frac{g_3^2}{g_1} > 0 \end{aligned} \qquad (7.25), (7.26)$$

These constraints (7.23)-(7.26) may be added to the nonlinear characteristic equations for the pole placement to guarantee the stability of the closed-loop poles.

We wish to assign the poles of the 3 dof system in Figure 7.1 using the symmetric fully populated control gains to the prescribed values,

$$\begin{aligned} \lambda_{1,2} &= -0.0100 \pm 0.7\mathrm{i} \\ \lambda_{3,4} &= -0.0600 \pm 1.8\mathrm{i} \end{aligned}$$

The control gains before adding the positive-definite constraints are found to be,

$$\mathbf{G} = \begin{bmatrix} 1.498 & -0.582 \\ -0.582 & 0.985 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 0.164 & 0.0205 \\ 0.0205 & -0.0054 \end{bmatrix}$$

The matrices $\mathbf{G}$ and $\mathbf{F}$ are not positive definite since;

$$\text{eig}(\mathbf{F}) = \begin{bmatrix} -0.22 \\ 0.4533 \end{bmatrix} \text{ and } \text{eig}(\mathbf{G}) = \begin{bmatrix} 0.54 \\ 2.07 \end{bmatrix}$$

Adding the four constraints (7.23)-(7.26) to the nonlinear characteristic equations, the control gains are obtained as,

$$\mathbf{G} = \begin{bmatrix} 2.076 & -0.222 \\ -0.222 & 0.546 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 0.0699 & -0.0248 \\ 0.0248 & -0.06777 \end{bmatrix}$$

which leads to the following closed-loop poles;

$$\lambda_{1,2} = -0.0100 \pm 0.7i$$
$$\lambda_{3,4} = -0.0600 \pm 1.8i$$
$$\lambda_{5,6} = -0.0504 \pm 2.34i$$

and the control gain matrices become positive definite.

$$\text{eig}(\mathbf{F}) = \begin{bmatrix} 0.043 \\ 0.0937 \end{bmatrix} \text{ and } \text{eig}(\mathbf{G}) = \begin{bmatrix} 0.514 \\ 2.1084 \end{bmatrix}$$

Therefore, the stability of the closed-loop poles may be guaranteed.

7.5 Inclusion of acceleration feedback

Besides active damping and active stiffness, active mass can also be achieved using acceleration feedback. Assignment of poles, zeros and simultaneous assignment of poles and zeros may be achieved by solving the characteristic equations for the unknown control gains. The feedback law is expressed as,

$$\mathbf{u}(s) = -\left(\mathbf{G} + s\mathbf{F} + s^2\mathbf{E}\right)\mathbf{y}(s) \tag{7.27}$$

so that the output and rate gains are given by the terms in the matrices $\mathbf{G}, \mathbf{F}, \mathbf{E} \in \Re^{m \times l}$. The control gains are considered to be diagonal allowing the assignment of three pairs of complex conjugate poles.

Example 7.6: Assignment of poles including acceleration feedback

The problem of pole assignment is considered for the 3dof system. We wish to assign the following poles;

$$\lambda_{1,2} = -0.0100 \pm 0.7\mathrm{i}$$
$$\lambda_{3,4} = -0.0600 \pm 1.4\mathrm{i}$$
$$\lambda_{5,6} = -0.06 \pm 2.35\mathrm{i}$$

The control gains are found to be;

$$\mathbf{E} = \begin{bmatrix} -1.0124 & 0 \\ 0 & 4.2767 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0.0446 & 0 \\ 0 & -0.2924 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} -3.2525 & 0 \\ 0 & 5.7509 \end{bmatrix}$$

Using the state-space method, the modified mass, stiffness and damping matrices are obtained as;

$$\tilde{\mathbf{M}} = \begin{bmatrix} 0.9876 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7.2767 \end{bmatrix}, \tilde{\mathbf{K}} = \begin{bmatrix} 2.7475 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 8.7509 \end{bmatrix}, \tilde{\mathbf{C}} = \begin{bmatrix} 0.1446 & 0 & 0 \\ 0 & 0.1 & -0.1 \\ 0 & -0.1 & -0.1924 \end{bmatrix}$$

The plot of modified receptance in Figure 7.6 indicates the assignment of poles at their prescribed locations.

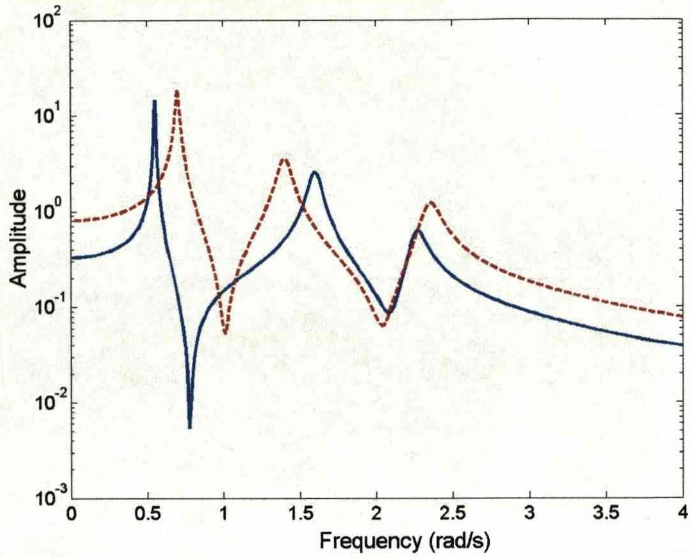


Figure 7.6. Initial receptance (solid line) and modified receptance (dashed line)

Another set of poles are considered for the pole assignment problem which all the poles are assigned to the lower frequencies.

$$\lambda_{1,2} = -0.0100 \pm 0.45i$$
$$\lambda_{3,4} = -0.0400 \pm 1.5i$$
$$\lambda_{5,6} = -0.06 \pm 2.1i$$

The control gains are found to be;

$$\mathbf{E} = \begin{bmatrix} -0.7348 & 0 \\ 0 & -1.6909 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} -1.8564 & 0 \\ 0 & 1.8745 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} -2.8440 & 0 \\ 0 & 0.2419 \end{bmatrix}$$

The plot of modified receptance in Figure 7.7 indicates the assignment of poles at their prescribed locations.

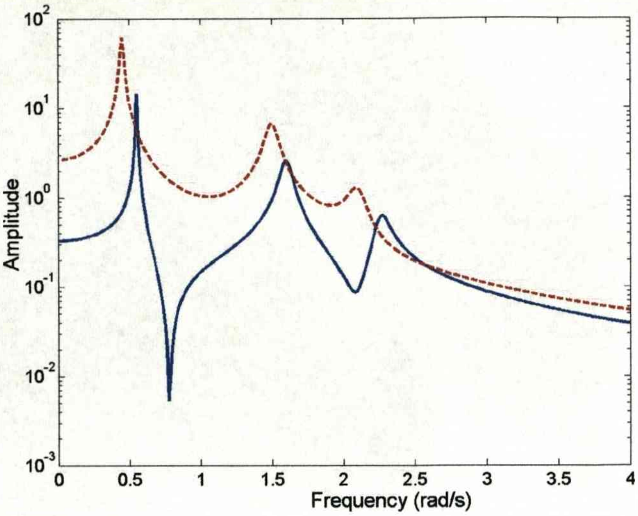


Figure 7.7. Initial receptance (solid line) and modified receptance (dashed line)

Using the state-space method, the modified mass, stiffness and damping matrices are obtained as;

$$\tilde{\mathbf{M}} = \begin{bmatrix} 1.2652 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.3091 \end{bmatrix}, \tilde{\mathbf{K}} = \begin{bmatrix} 3.1560 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3.2419 \end{bmatrix}, \tilde{\mathbf{C}} = \begin{bmatrix} -1.7564 & 0 & 0 \\ 0 & 0.1 & -0.1 \\ 0 & -0.1 & 1.9745 \end{bmatrix}$$

which leads to the prescribed poles.

7.6 Output feedback with time delay

One of the important parameters which limit the performance of feedback controllers is time delay. Time delays may arise from digital implementation of the controller. Time delays can induce unmodelled phase shifts to the system. Such phase shifts are undesirable and may result in unstable closed-loop response. It was shown [F5] that if either displacement or acceleration feedback are implemented with delay in a control loop, the effective damping may be reduced so that the system becomes unstable. However, velocity feedback with time delay was seen to be a more robust control strategy than the displacement and acceleration feedback [F5]. In this section the effect of time delay is considered for the problem of eigenvalue assignment. Equations with time delay may be written as,

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{B}\mathbf{u}(t-\Delta)+\mathbf{p}(t) \tag{7.28}$$

where Δ represents the time delay. Likewise, the output equation may be written as,

$$\mathbf{y}(t-\Delta)=\mathbf{D}\mathbf{x}(t-\Delta), \quad \mathbf{D}=\mathbf{B}^T \tag{7.29}$$

where $\mathbf{D}\in\Re^{l\times n}$ is the sensor distribution matrix and the output is denoted by $\mathbf{y}\in\Re^{l\times n}$. The feedback law is expressed as,

$$\mathbf{u}(t-\Delta)=-\mathbf{G}\mathbf{y}(t-\Delta)-\mathbf{F}\dot{\mathbf{y}}(t-\Delta) \tag{7.30}$$

Combining equations (7.28)-(7.30),

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=-\mathbf{B}\mathbf{G}\mathbf{B}^T\mathbf{x}(t-\Delta)-\mathbf{B}\mathbf{F}\mathbf{B}^T\dot{\mathbf{x}}(t-\Delta)+\mathbf{p}(t) \tag{7.31}$$

and by the Taylor expansion,

$$\begin{aligned}\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=&-\mathbf{B}\mathbf{G}\mathbf{B}^T\mathbf{x}(t)+\mathbf{B}\Delta\mathbf{G}\mathbf{B}^T\dot{\mathbf{x}}(t)\\ &-\mathbf{B}\mathbf{F}\mathbf{B}^T\dot{\mathbf{x}}(t)+\mathbf{B}\Delta\mathbf{F}\mathbf{B}^T\ddot{\mathbf{x}}(t)+\mathbf{p}(t)\end{aligned} \tag{7.32}$$

In the frequency domain,

$$\left(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K}+\mathbf{Z}(s)\right)\mathbf{x}(s)=\mathbf{p}(s) \tag{7.33}$$

where

$$\mathbf{Z}(s) = -\mathbf{B}\left(\Delta\mathbf{F}s^2 + (\Delta\mathbf{G} - \mathbf{F})s - \mathbf{G}\right)\mathbf{B}^T \tag{7.34}$$

Pre-multiplying both sides of equation (7.33) by,

$$\mathbf{H}(s) = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right) \tag{7.35}$$

leads to an expression for the closed-loop receptance matrix as,

$$\hat{\mathbf{H}}(s) = \frac{\operatorname{adj}(\mathbf{I} + \mathbf{H}(s)\mathbf{Z}(s, \mathbf{G}, \mathbf{F}))\mathbf{H}(s)}{\det(\mathrm{I} + \mathbf{H}(s)\mathbf{Z}(s, \mathbf{G}, \mathbf{F}))} \tag{7.36}$$

Poles may be assigned by choosing terms in **G**, **F** that cause the determinant in the denominator to vanish. Zeros may be assigned by choosing terms in **G**, **F** that cause the matrix product in the numerator to vanish.

Example 7.7: Assignment of poles with time delay

For the system described by,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Delta = 0.02$$

We wish to assign the following poles,

$$\lambda_{1,2} = -0.1 \pm 2\mathrm{i}$$
$$\lambda_{3,4} = -0.05 \pm 5\mathrm{i}$$

For purposes of simplification we consider the control gains to be diagonal,

$$\mathbf{G} = \operatorname{diag}(g_i), \mathbf{F} = \operatorname{diag}(f_i), i = 1,2.$$

and solve four nonlinear equations in control gains,

$$\det\left(\mathrm{I} - \mathbf{H}(\lambda_j)\mathbf{B}\left(\Delta\mathbf{F}\lambda_j^2 + (\Delta\mathbf{G} - \mathbf{F})\lambda_j - \mathbf{G}\right)\mathbf{B}^T\right) = 0; \quad j = 1, \ldots, 4$$

The solution is found to be:

$\mathbf{F} = \text{diag}(-0.2572, 0.5325), \mathbf{G} = \text{diag}(2.0789, 21.6888)$

such that the necessary gains are real.

Solution of the state-space eigenvalue problem with state matrix,

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\left(\mathbf{M} - \mathbf{B}\Delta\mathbf{F}\mathbf{B}^T\right)^{-1}\left(\mathbf{K} + \mathbf{B}\mathbf{G}\mathbf{B}^T\right) & -\left(\mathbf{M} - \mathbf{B}\Delta\mathbf{F}\mathbf{B}^T\right)^{-1}\left(\mathbf{C} - \mathbf{B}\Delta\mathbf{G}\mathbf{B}^T + \mathbf{B}\mathbf{F}\mathbf{B}^T\right) \end{bmatrix} \quad (7.37)$$

and terms as given above returns the desired set of eigenvalues,

$$\lambda_{1,2} = -0.1 \pm 2i$$
$$\lambda_{3,4} = -0.05 \pm 5i$$

7.7 Conclusion

Active vibration suppression by eigenvalue assignment using output feedback control is presented. Receptances of the open-loop system are used to assign poles and zeros instead of the usual $\mathbf{K}, \mathbf{C}, \mathbf{M}$ matrices. Collocated actuators and sensors are considered in the output feedback control, having practical advantages of avoiding instability due to spillover. The use of diagonal gain matrices results in a symmetric dynamic stiffness matrix and therefore maintains the interlacing property of poles and zeros. It is shown that one can assign more poles and zeros than the number of actuators and sensors when the gain matrices are fully populated. The stability of the closed-loop poles is guaranteed by constraining the fully populated control gain matrices to be positive definite. The characteristic equations, being nonlinear in the control gains, are used for the assignment of poles and zeros and simultaneous assignment of poles and zeros. The acceleration feedback control is also added to the displacement and velocity feedback. The characteristic equations for the pole-zero assignment are reformulated with the time delay present. The effect of time-delay on the output feedback is significant and may result in an unstable closed-loop system if neglected. Numerical examples were presented to demonstrate the receptance method for the assignment of poles and zeros (natural frequencies and antiresonances) using collocated actuators and sensors.

Chapter 8

EXPERIMENTS IN ACTIVE VIBRATION CONTROL

8.0 Introduction

In this chapter physical experiments for active vibration control by eigenvalue assignment using the output-feedback receptance method are described [M16]. Experiments are carried out on a T-shaped plate using collocated inertial actuators and sensors. Receptances $\mathbf{H}(s)$ are obtained by fitting rational fraction polynomials to the measured receptances $\mathbf{H}(\mathrm{i}\omega)$. Nonlinear characteristic equations for pole-zero assignment are solved to determine the displacement and velocity feedback gains. Results of the measured closed-loop receptances show peaks and dips that agree very closely with the imaginary parts of the assigned poles and zeros. Robustness of the closed-loop response is analysed using the concept of singular values of the return difference transfer matrix. Experimental results demonstrate the improvement of stability robustness in the closed-loop receptances.

8.1 Experiments on a T-shaped plate

There has been a great deal of theoretical work in the area of active vibration control; however not much experimental work has been dedicated to the problem of pole and zero assignment. Most experiments have concentrated on active damping using velocity feedback. Electronic damping was considered by Swigert and Forward in a two-part paper, theory [S9] and experiment [F2]. A high level of damping was achieved in the first two orthogonal bending modes of a cylindrical mast.

Active damping by velocity feedback was considered by Bianchi and Gardonio *et al.* in [B10] for the control of sound transmission of a panel. The panel comprising 16 decentralised units each with a collocated accelerometer-sensor and a piezoceramic patch actuator with a single channel velocity feedback controller generates active damping. Preumont *et al.* [P4] considered active damping by a local force feedback using piezoelectric actuators and achieved a significant increase of damping in the vibration modes. The development and application of smart structures and materials (adaptive structures) to spacecraft for dynamic control is described in [D1].

Experiments based on independent modal space control theory developed by Meirovitch [M6] have encountered practical difficulties. To control the modes independently and to avoid control and observation spillover a large number of actuators and sensors may be required. Also the model of damping is assumed to be of the proportional type in order to decouple the dynamic equations. However, there are different forms of damping such as viscous, material, friction, impact etc present in a physical structure and cannot be modelled accurately. Stobener and Gaul [S8] applied the method of independent modal space control to control the vibration modes of a car body. In their approach the modal data (frequencies and mode-shapes) were extracted from the measured receptances and the damping matrix was assumed to be proportional.

Experiments presented in this chapter are based on the theory of the receptance method to assign poles and zeros separately and for the simultaneous assignment of poles and zeros. The receptance transfer function $\mathbf{H}(s)$ is obtained from the measured receptance frequency response function $\mathbf{H}(\mathrm{i}\omega)$ and consequently there is no requirement to evaluate the system matrices $\mathbf{K}, \mathbf{C}, \mathbf{M}$. Experiments were carried out on the T-shaped plate shown in Figure 8.1 using two sets of collocated sensors (Kistler accelerometer type 8636C50) and inertial actuators (Micromega Dynamics type IA-01). The internal active damper that forms part of the inertial actuator, based on analogue integration of acceleration was not used. Instead, signals from the Kistler accelerometers were integrated twice by digital means, thereby enabling velocity and displacement feedback, implemented using MATLAB/Simulink and dSPACE. Open-loop receptances were measured by means of a modal test using hammer excitation with the inertial actuators in place but not operational. The first two natural frequencies of the open-loop system were at 40 Hz (252 rad/s) and 52 Hz (325 rad/s). The first mode was a stem bending mode and the second showed stem twisting with the two arms in anti-phase. The third natural frequency, in-phase arm bending, occurred at 125 Hz (785 rad/s) well away from the first two modes. The first two mode shapes are shown in Figure 8.2.

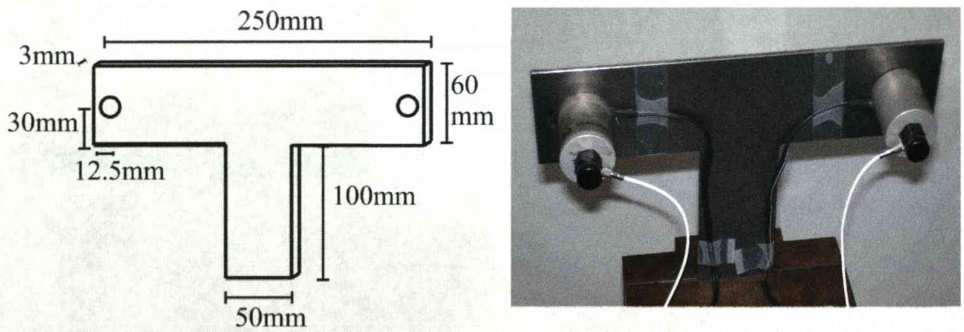


Figure 8.1.T-shaped plate (a) dimensions, (b) experimental arrangement

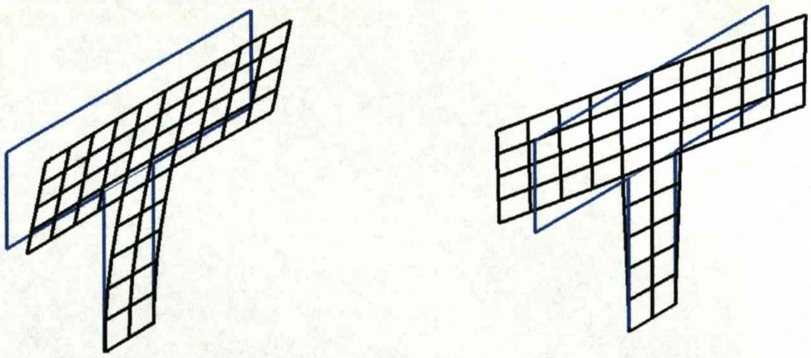


Figure 8.2. (a) First mode (stem bending), (b) Second mode (stem twisting)

The purpose of using receptances instead of the $\mathbf{K}, \mathbf{C}, \mathbf{M}$ matrices is that uncertainties, approximations and assumptions associated with finite element models are avoided completely. Also there is no need for model reduction or the use of observers to estimate the unmeasured states because the system equations are complete with just a small number of states. However $\mathbf{H}(\mathrm{i}\omega)$ is available from vibration experiments and not $\mathbf{H}(s)$, as required by the theory. Rational fraction polynomials were fitted to the measured $\mathbf{H}(\mathrm{i}\omega)$ and the coefficients of the numerator and denominator polynomials determined [F1]. The coefficients were found by solving a least-squares problem, which should be well conditioned so that the coefficients are not sensitive to small changes in the measurements. The receptances $h_{11}(\mathrm{i}\omega)$ and $h_{22}(\mathrm{i}\omega)$ were almost identical because of geometric symmetry of the T-plate and due to linearity $h_{12}(\mathrm{i}\omega)$ and $h_{21}(\mathrm{i}\omega)$ were found to be very similar. The H1 estimator was applied using the following test parameters: sample rate 256 Hz, frequency resolution 0.125 Hz, number of impacts 20 and an exponential window with a decay to 1% applied to the measured accelerations.

The fitted receptances are presented in Figure 8.3 where measurements are represented by full lines (blue) and fitted curves are shown as dashed lines (red). The good agreement shown in Figure 8.3, for amplitude, was similarly obtained for phase.

The following polynomials were identified,

$$h_{11}(s)=h_{22}(s)=\frac{7.956\times10^{-10}s^2+1.382\times10^{-8}s+6.438\times10^{-5}}{1.476\times10^{-10}s^4+4.987\times10^{-9}s^3+2.515\times10^{-5}s^2+0.0003862s+1}$$

$$h_{12}(s)=h_{21}(s)=\frac{-1.334\times10^{-10}s^2-4.678\times10^{-9}s+5.208\times10^{-6}}{1.476\times10^{-10}s^4+4.987\times10^{-9}s^3+2.515\times10^{-5}s^2+0.0003862\,s+1}$$

It should be pointed out that the rational fraction polynomials represent a model of the system. However all that is required of this model is that it is accurate at the chosen location of the poles and zeros to be assigned. In the case of lightly damped systems it is likely that the identified rational fraction polynomial will be accurate if the curve fit agrees closely with the measured terms in $\mathbf{H}(\mathrm{i}\omega)$.

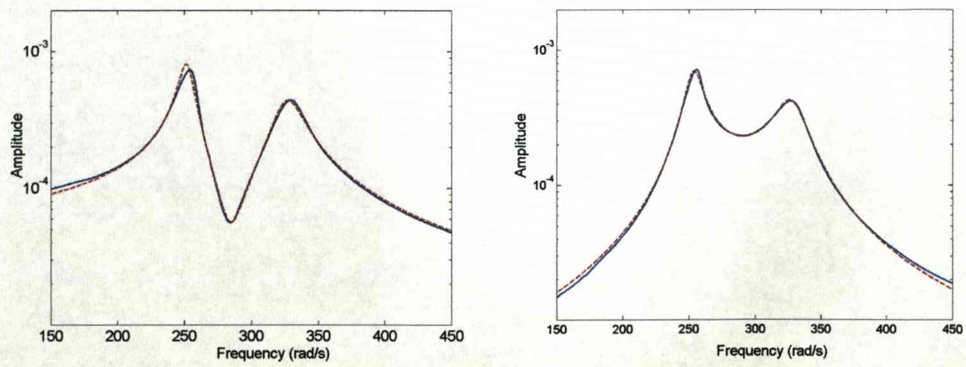


Figure 8.3. Rational fraction curve fit (a) $h_{11}(\mathrm{i}\omega)$, (b) $h_{12}(\mathrm{i}\omega)$ measurement – solid line; fitted curve – dashed line

The force-voltage transfer function of an inertial actuator with a fixed base may be expressed as [P2];

$$\frac{\mathrm{f}}{\mathrm{V}} = \mathrm{g}\frac{s^2}{s^2 + 2\zeta_p \omega_p s + \omega_p^2} \tag{8.1}$$

Above a critical frequency it is shown that the actuator behaves as an ideal force generator with constant amplitude and zero phase. This is the case shown in Figure 8.4, the force being measured by a sensor inserted between the base of the actuator and a heavy rigid mass. Preumont's analysis is in fact a simplification of the real situation since the actuator was fixed to the flexible T-plate in our control application. However a small distance away from the T-plate resonances the expression was found to hold good. A gain of $g = 0.37$ was obtained from the curve shown in Figure 8.4, being the average value of the almost constant pure gain (zero phase) at frequencies greater than the natural frequency of the actuator at around 55 rad/sec.

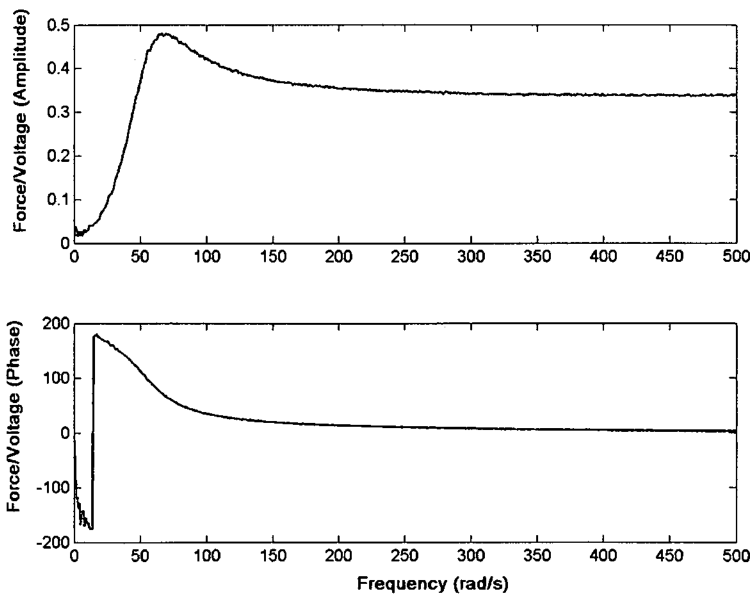


Figure 8.4. Actuator force-voltage transfer function.

8.1.1 Experiment: Assignment of poles

Poles were firstly assigned at $\lambda_{1,2} = -12 \pm 284\text{i}$ and $\lambda_{3,4} = -22 \pm 365\text{i}$ and then in a second test at $\lambda_{1,2} = -10 \pm 290\text{i}$ and $\lambda_{3,4} = -25 \pm 375\text{i}$. In this and subsequent experiments the force and sensor distribution matrices were set to $\mathbf{B} = \mathbf{D}^T = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix. In the first test, gains with the values of $\mathbf{G} = \text{diag}(205, 10955)$ and $\mathbf{F} = \text{diag}(4.4, 11.8)$ were found and in the second test $\mathbf{G} = \text{diag}(124, 14339)$ and $\mathbf{F} = \text{diag}(0.5, 11.19)$. Figure 8.5 shows experimental receptances for the open-loop system as the full line (blue) and closed-loop receptances, h_{11} for the first and second tests. The closed-loop receptance for the first test is shown as a dashed line (red) and for the second test as a dot-dashed line (green). As expected, the peaks of the dashed and dot-dashed lines can be seen to agree very well with the imaginary parts of the assigned poles for the first and second tests respectively. The actuators were operational during the closed-loop modal tests used to determine the receptances from excitation by an instrumented hammer.

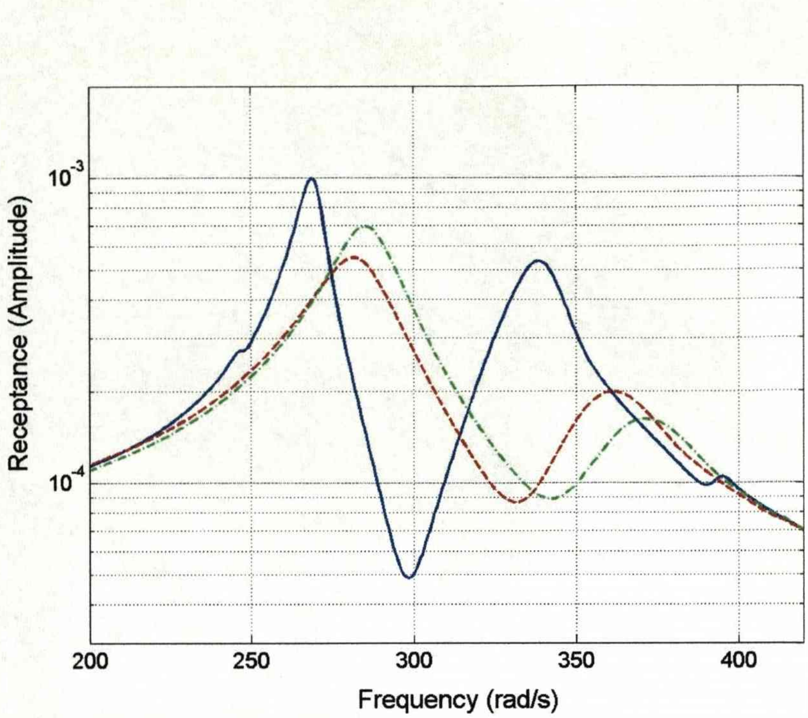


Figure 8.5. Assignment of poles
solid line: open-loop receptance
dashed line: closed-loop receptance for the first test
dot-dashed line: closed-loop receptance for the second test

8.1.2 Experiment: Assignment of zeros

To maintain geometrical symmetry of the T-plate identical zeros were assigned to h_{11} and h_{22} at $\mu_1, \mu_2 = -10 \pm 300i$. Gains were determined as $\mathbf{F} = 1.8\mathbf{I}, \mathbf{G} = 3735\mathbf{I}$. Experimental open-loop and closed-loop receptances are shown in Figure 8.6 where the frequency of the zeros is seen to agree very well with the assigned value of 300 rad/s and the poles remain stable.

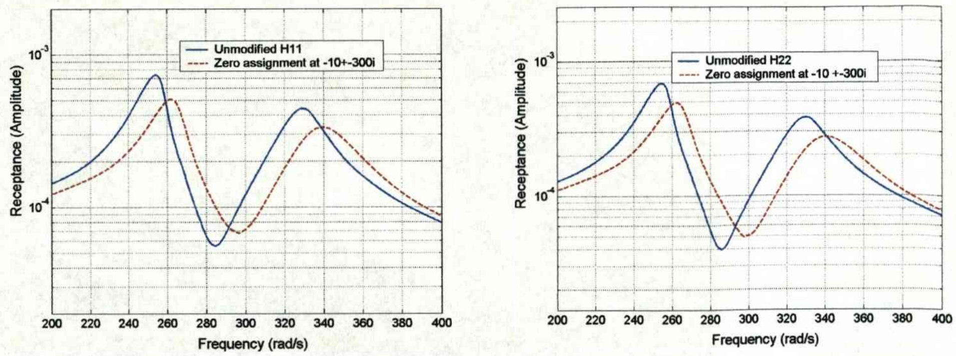


Figure 8.6. Assignment of zeros (a) h_{11}, (b) h_{22}

8.1.3 Experiment: Simultaneous assignment of poles and zeros

Poles and zeros were assigned to h_{11} at $\lambda_{1,2} = -15 \pm 295\mathrm{i}$ and $\mu_{1,2} = -12 \pm 265\mathrm{i}$ respectively. Gain values of $\mathbf{G} = \mathrm{diag}(3720, 2355)$ and $\mathbf{F} = \mathrm{diag}(2.24, 5.75)$ were found and the resulting open-loop and closed loop receptances were plotted as shown in Figure 8.7. It can be seen in Figure 8.7 that the frequencies of the first peak and dip of the dashed (red) line, that represents the closed-loop receptance, agree closely with the imaginary part of the assigned poles and zeros.

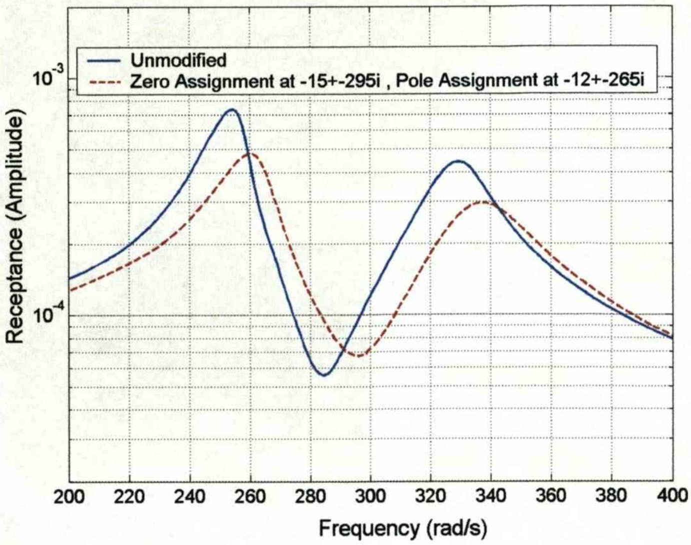


Figure 8.7. Simultaneous assignment of poles and zeros

8.2 Stability robustness

The robustness of feedback control systems with respect to stability has long been recognized. In classical control the robustness of the single-input single-output (SISO) control system is well understood by various graphical means such as Bode, Nyquist and Nichols plots. From these plots one can determine the minimum change in the model frequency response that leads to instability. Gain and phase margins are the common measures of stability for an SISO system. In multi-input multi-output (MIMO) systems the stability margins should be satisfied in all its loops simultaneously. Therefore the robustness of a MIMO system is characterised in terms of the minimum singular value (MSV) of its return difference transfer function matrix. The concept of MSV as the measure of robustness was discussed in detail by Maciejowski [M3].

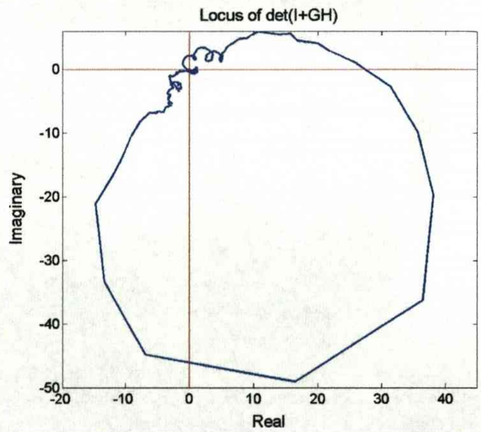
The inverse problem of pole assignment for the multi-input state feedback control as indicated by Ram and Elhay [R5] may have a family of solutions for the control gain. Therefore, defining solutions in which the assigned poles are as insensitive to perturbations as possible is the aim of robustness analysis. It is known [W2] that the sensitivities of the eigenvalues of a matrix are dependent on the corresponding eigenvectors. Kautsky *et al.* [K3] introduced four iterative algorithms which compute robust solutions to the multi-input state feedback pole assignment problem. All the algorithms use the concept of orthogonal projection into linear subspaces of eigenvectors to improve iteratively various equivalent measures of conditioning for robustness of the closed-loop system. Juang *et al.* [J1] presented a non-iterative algorithm based on the technique of Kautsky *et al* [K3] for output feedback control which uses design freedom to choose the minimum control gain when the number of assigned eigenvalues is less than the number of assignable eigenvalues. Kautsky and Nichols [K4] presented an efficient and reliable numerical method for minimizing the pole sensitivity to structured perturbations. It was demonstrated that after a small number of iterations the improvement of the sensitivity measure is achieved when the system is subjected to random perturbations.

8.2.1 Stability robustness for the T-shaped plate

In this section the stability robustness for the experimental T-plate structure to active vibration control by pole-zero placement is addressed. The problem in Section 8.1.2 is considered, namely the assignment of zeros of h_{11} and h_{22} to $\mu_1, \mu_2 = -10 \pm 300\mathrm{i}$. The open-loop transfer function between the input voltage to the actuators and the output displacement $\mathbf{y}(s)$ is then defined as $0.37 \times \mathbf{H}(s)\mathbf{B}\mathrm{diag}(g_i + sf_i)\mathbf{B}^T$, where the gain of 0.37 represents the actuator dynamic as described previously.

Open-loop experiments were carried out by applying random voltages to the actuators in the range of 0-1000Hz and measuring the output voltages, calibrated for displacement, from the dSPACE board. The frequency-domain requirements for the stability of single-input, single-output (SISO) systems take the form of the standard Nyquist criterion and in the multiple-input, multiple-output (MIMO) case they involve its multivariable generalisation. Thus the system may be considered stable [R9] if $\det\left[\mathbf{I} + 0.37 \times \mathbf{H}(\mathrm{i}\omega)\mathbf{B}\,\mathrm{diag}(g_i + \mathrm{i}\omega f_i)\mathbf{B}^T\right]$, $0 < \omega < \infty$, does not enclose the point $(0, 0)$. It can bee seen from Figures 12(a) and 12(b) that this appears to be the case, although marginal stability is approached extremely closely. The spectrum in Figure 8.9 shows good roll-off of the eigenvalues of the open-loop transfer function thereby confirming that in practice spillover does not lead to instability at higher frequencies.

The polar plot shown in Figures 8.8(a) and 8.8(b) does not encircle point (0, 0), and therefore the closed-loop system is deemed stable. The spectrum in Figure 8.9 shows good roll-off of the eigenvalues of the open-loop transfer function so that spillover does not lead to instability at higher frequencies.



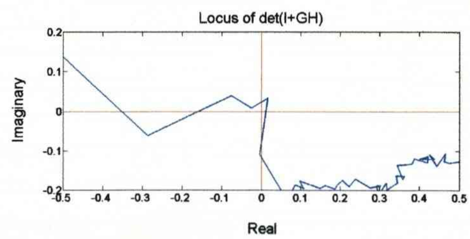


Figure 8.8. Polar plot of $\det\left[\mathbf{I} + 0.37 \times \mathbf{H}(\mathrm{i}\omega)\mathbf{B}\,diag(g_i + \mathrm{i}\omega f_i)\mathbf{B}^T\right]$

(a) Full curve; (b) Magnified view close to the origin

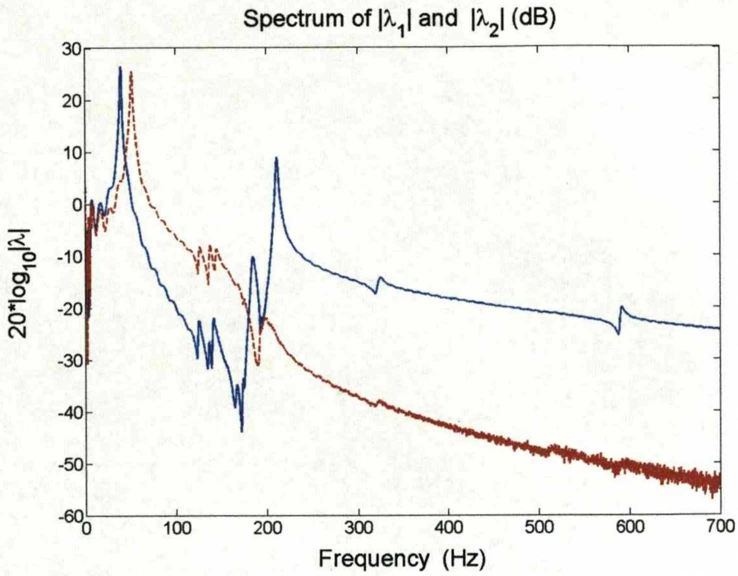


Figure 8.9. Spectrum of eigenvalues λ_1 (blue solid line) and λ_2 (red dashed line) of the open-loop transfer function $0.37 \times \mathbf{H}(\mathrm{i}\omega)\mathbf{B}\mathrm{diag}(g_i + \mathrm{i}\omega f_i)\mathbf{B}^T$

The robustness of the closed-loop system can be improved by increasing the minimum singular value $\underline{\sigma}$ of $\left[\mathbf{I} + 0.37 \times \mathbf{H}(s)\mathbf{B}\mathrm{diag}(g_i + sf_i)\mathbf{B}^T\right]$ as explained by Maciejowski [M3], thereby defining the following constraint,

$$\underline{\sigma}\left[\mathbf{I} + 0.37 \times \mathbf{H}(s)\mathbf{B}\mathrm{diag}(g_i + sf_i)\mathbf{B}^T\right] > r \tag{8.2}$$

on the solution of the nonlinear characteristic equations.

In our particular example, a constraint of $\underline{\sigma} > 0.7$ was applied over the frequency range of 50-100Hz. The resulting control gains were found to be,

$$\mathbf{F} = \begin{bmatrix} 12.7 & 0 \\ 0 & 13.8 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 3734.9 & 0 \\ 0 & 3734.9 \end{bmatrix}$$

In Figure 8.10(a) the solid line denotes $\underline{\sigma}$ versus frequency without the constraint. The dashed line is the case of $\underline{\sigma} > 0.7$ for the range of 50-

100Hz. A series of modal tests using hammer excitation were carried out with the feedback control gains obtained with the added constraints $\underline{\sigma} > 0.6$ and $\underline{\sigma} > 0.7$. The solid line in Figure 8.10(b) represents the unconstrained open-loop receptance. The dashed and dot-dashed lines represent the closed-loop receptances after the added constraints $\underline{\sigma} > 0.6$ and $\underline{\sigma} > 0.7$ were applied over the range of 50-100 Hz. It is seen that the effect of the constraints is to add damping and thereby improve the stability robustness of the T-plate system. Of course, this is achieved at the cost of reduced accuracy in the placement of the zeros.

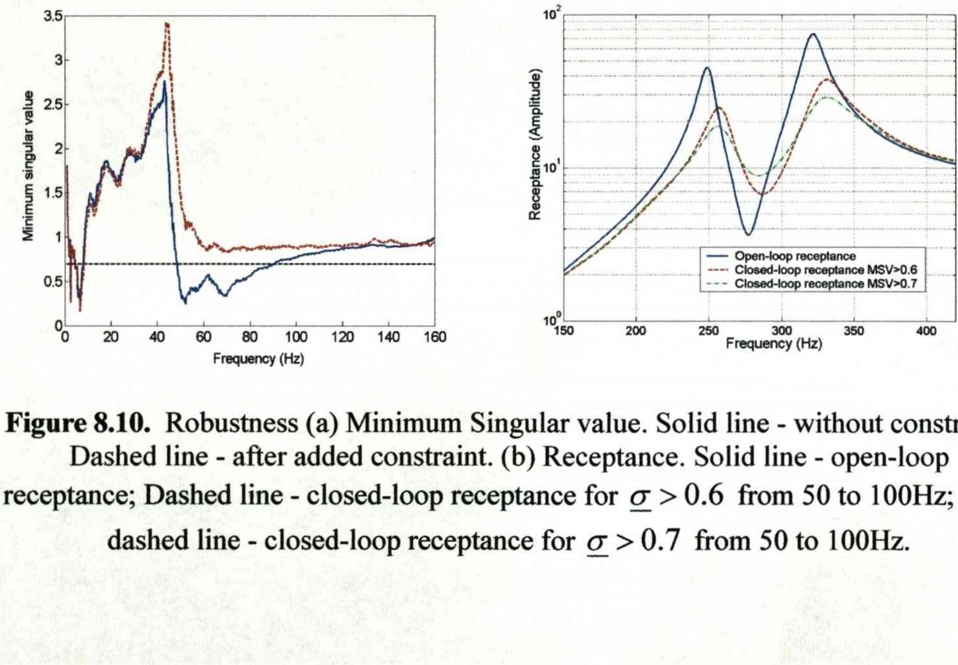


Figure 8.10. Robustness (a) Minimum Singular value. Solid line - without constraint; Dashed line - after added constraint. (b) Receptance. Solid line - open-loop receptance; Dashed line - closed-loop receptance for $\underline{\sigma} > 0.6$ from 50 to 100Hz; Dot-dashed line - closed-loop receptance for $\underline{\sigma} > 0.7$ from 50 to 100Hz.

A further constraint, $\underline{\sigma} > 0.45$, was then added to the range 6-10 Hz to address the sharp 'dip' in the dashed line of Figure 8.10(a) at low frequencies. In the range 50-100Hz the lowest singular value was constrained such that $\underline{\sigma} > 0.9$. Control gains were then obtained as,

$$\mathbf{F} = \begin{bmatrix} 17.35 & 0 \\ 0 & 22.64 \end{bmatrix} \mathbf{G} = \begin{bmatrix} 1709.5 & 0 \\ 0 & 2164 \end{bmatrix}$$

Figure 8.11 shows the minimum singular values and the closed-loop receptances for the system, now with two constraints. This results in considerably increased damping over the previous test and further deterioration in the accuracy of placing the zeros.

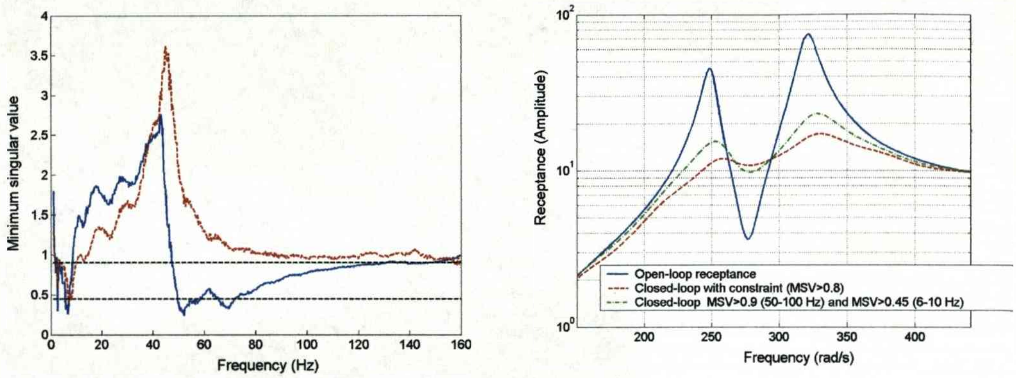


Figure 8.11. Robustness (a) Minimum Singular value. Solid line - without constraint; Dashed line - after added constraint, (b) Receptance. Solid line - open-loop receptance; Dashed line - closed-loop receptance for $\underline{\sigma} > 0.8$ from 50 to 100Hz; Dot-dashed line - closed-loop receptance for $\underline{\sigma} > 0.9$ from 50 to 100Hz and $\underline{\sigma} > 0.45$ from 6 to 10 Hz.

8.3 Conclusion

This chapter presents experimental results carried out on a T-shaped plate for eigenvalue assignment using output feedback control. Collocated inertial actuators and accelerometers were used on the T-shaped plate to assign poles and zeros (natural frequencies and antiresonances). In the experiments, receptances $\mathbf{H}(s)$ were determined from the measured $\mathbf{H}(\mathrm{i}\omega)$ by fitting rational fraction polynomials. Closed-loop receptances, obtained by applying gains determined from the analysis, agreed very closely with the expected receptances having the assigned poles and zeros. The robustness of control systems with respect to model uncertainty was achieved by constructing the minimum singular values of the return difference transfer function matrix. This was found to result in higher damping in the closed-loop response.

Chapter 9

EIGENVALUE SENSITIVITY FROM RECEPTANCE MEASUREMENTS

9.0 Introduction

In the inverse problem of eigenvalue assignment the sensitivities of the eigenvalues with respect to the modification parameter plays an important role in the design and analysis of dynamic systems. In active vibration control it might be for example desirable to design a controller that assigns the prescribed eigenvalues with the least control effort whilst the remaining unassigned eigenvalues are made insensitive.

The basic method for evaluating the sensitivities of the eigenvalues of the problem $\mathbf{Ax} = \lambda\mathbf{x}$ goes back to the work by Fox and Kapoor [F3] who utilized the right and left eigenvectors to calculate the eigenvalue sensitivities. Rogers [R8] presented eigenvalue and eigenvector sensitivities for the general first-order matrix pencil, including the cases of general (non-proportional) damping and gyroscopic damping, suitable therefore for the case of asymmetric matrices. Nelson's method [N3] for finding the sensitivity of the jth left and right eigenvectors required only the same two eigenvectors in the calculation. It was pointed out that Nelson's method cannot be applied to determine the sensitivities of repeated eigenvalues [O1, P3, V1]. The sensitivity of a repeated eigenvalue of conservative vibrating systems has been determined by Ojalvo [O1] who utilized an uxiliary eigenvalue problem associated with the subspace spanned by the eigenvectors corresponding to the repeated eigenvalues. Prells and Friswell [P3] derived the sensitivity of a repeated eigenvalue from the characteristic equation without using the eigenvector data. Vessel and Ram *et al.* [V1] extended the method by Ojalvo to viscously damped vibratory systems and determined the sensitivity of a repeated eigenvalue of multiplicity $p \geq 2$. Mottershead [M8] showed the sensitivities of the zeros to be given by a linear combination of the sensitivities of the poles and the sensitivities of the eigenvectors.

In this chapter the sensitivity of the eigenvalues with respect to the control gains are determined from the matrix of measured receptances. Single-input state feedback is considered, resulting in a unit rank modification to the stiffness and damping matrices of the closed-loop system. It is demonstrated using numerical examples how eigenvalue sensitivities may be assigned so that selected eigenvalues are rendered sensitive, while other become insensitive.

9.1 Theory

The general second-order matrix differential equation is considered.

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(\mathrm{t}) \tag{9.1}$$

$$\mathbf{M}, \mathbf{C}, \mathbf{K} \in \Re^{n \times n}, \mathbf{M} = \mathbf{M}^T, \quad \mathbf{C} = \mathbf{C}^T, \quad \mathbf{K} = \mathbf{K}^T$$

$$\mathbf{v}^T\mathbf{M}\mathbf{v} > 0, \quad \mathbf{v}^T\mathbf{C}\mathbf{v} \geq 0, \quad \mathbf{v}^T\mathbf{K}\mathbf{v} \geq 0 \text{ for arbitrary } \mathbf{v} \neq \mathbf{0}, \quad \mathbf{v}, \mathbf{x} \in \Re^{n \times 1}.$$

The dynamics of this system is governed by the second-order matrix pencil,

$$P(s) = s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} \tag{9.2}$$

the eigenvalues and eigenvectors of which satisfy the following equation,

$$P(\sigma_j)\mathbf{v}_j = \mathbf{0} \tag{9.3}$$

These eigenvalues and eigenvectors may be arranged to form the spectral and modal matrices,

$$\mathbf{\Sigma} = \operatorname{diag}\{\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_{2n}\} \in \mathbb{C}^{2n\times 2n} \tag{9.4}$$

$$\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_{2n}] \in \mathbb{C}^{n\times 2n} \tag{9.5}$$

From Duncan's symmetric state-space arrangement it is readily shown that,

$$[\mathbf{\Sigma V}^T \ \mathbf{V}^T]\left(s\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}+\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\right)\begin{pmatrix}\mathbf{V\Sigma}\\ \mathbf{V}\end{pmatrix} = s\mathbf{I} - \mathbf{\Sigma} \tag{9.6}$$

where $\mathbf{V}$ is normalised so that,

$$\mathbf{V}^T\mathbf{MV\Sigma} + \mathbf{\Sigma V}^T\mathbf{MV} + \mathbf{V}^T\mathbf{CV} = \mathbf{I} \tag{9.7}$$

This equation was originally derived by Lancaster [L1] and subsequently by Datta *et al.* [D3].

Inversion of equation (9.6) leads to,

$$\left(s\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}+\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\right)^{-1} = \begin{pmatrix}\mathbf{V\Sigma}\\ \mathbf{V}\end{pmatrix}(s\mathbf{I}-\mathbf{\Sigma})^{-1}[\mathbf{\Sigma V}^T \ \mathbf{V}^T] \tag{9.8}$$

and by combining the two matrices in the left-hand-side bracket,

$$\left(s\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}+\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\right) = \begin{bmatrix}-\mathbf{M} & s\mathbf{M}\\ s\mathbf{M} & s\mathbf{C}+\mathbf{K}\end{bmatrix} \tag{9.9}$$

the inverse matrix is found as,

$$\begin{bmatrix}-\mathbf{M} & s\mathbf{M}\\ s\mathbf{M} & s\mathbf{C}+\mathbf{K}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{M}^{-1}+s^2(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K})^{-1} & s(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K})^{-1}\\ s(s^2\mathbf{M}+s\mathbf{C}+\mathbf{K})^{-1} & (s^2\mathbf{M}+s\mathbf{C}+\mathbf{K})^{-1}\end{bmatrix} \tag{9.10}$$

The receptance matrix is obtained from the terms in the bottom right-hand corners of the right-hand sides of equations (9.8) and (9.10),

$$\mathbf{H}(s) = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1} = \mathbf{V}(s\mathbf{I} - \mathbf{\Sigma})^{-1}\mathbf{V}^T \tag{9.11}$$

resulting in the well-know expression [H2],

$$\mathbf{H}(s) = \mathbf{V}\frac{\mathrm{adj}(s\mathbf{I} - \mathbf{\Sigma})}{\det(s\mathbf{I} - \mathbf{\Sigma})}\mathbf{V}^T = \sum_{k=1}^{2n}\frac{\mathbf{v}_k\mathbf{v}_k^T}{(s - \sigma_k)} \tag{9.12}$$

since $\mathrm{adj}(s\mathbf{I} - \mathbf{\Sigma}) = \mathrm{diag}\left(\det(s\mathbf{I} - \mathbf{\Sigma})_k\right) \in \mathbb{C}^{n\times n}$ where the matrix subscript k denotes the elimination of the k^{th} row and column.

Alternatively $\mathbf{H}(s)$ may be expressed as,

$$\mathbf{H}(s) = \left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]^{-1} = \frac{\mathrm{adj}\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]}{\det\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]} = \frac{\mathbf{N}(s)}{d(s)} \tag{9.13}$$

We will return to these expressions (9.12) and (9.13) in the analysis that follows.

9.2 Sensitivity from the receptance matrix

The poles are given by the roots of the function,

$$p(s) = 1 + (\mathbf{g} + s\mathbf{f})^T\mathbf{H}(s)\mathbf{b} \tag{9.14}$$

so that by combining equations (9.13) and (9.14),

$$q(s) = d(s) + (\mathbf{g} + s\mathbf{f})^T\mathbf{N}(s)\mathbf{b} \tag{9.15}$$

At the jth eigenvalue of the closed-loop system,

$$d(\lambda_j) + (\mathbf{g} + \lambda_j\mathbf{f})^T\mathbf{N}(\lambda_j)\mathbf{b} = 0 \tag{9.16}$$

Consider a small change to the ith term in $\mathbf{g}$, denoted $\delta g \times \mathbf{e}_i$ so that the new poles become $\lambda_j + \delta\lambda_j$. Then,

$$d\left(\lambda_j + \delta\lambda_j\right) = d\left(\lambda_j\right) + \left.\frac{\partial d}{\partial s}\right|_{s=\lambda_j} \frac{\partial \lambda_j}{\partial g}\delta g \tag{9.17}$$

and,

$$\mathbf{N}\left(\lambda_j + \delta\lambda_j\right) = \mathbf{N}\left(\lambda_j\right) + \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_j} \frac{\partial \lambda_j}{\partial g}\delta g \tag{9.18}$$

The poles of the perturbed system are then given by solving,

$$d\left(\lambda_j + \delta\lambda_j\right) + \left(\mathbf{g} + \delta g \times \mathbf{e}_i + \left(\lambda_j + \partial\lambda_j\right)\mathbf{f}\right)^T \mathbf{N}\left(\lambda_j + \delta\lambda_j\right)\mathbf{b} = 0 \tag{9.19}$$

which can be expanded to give,

$$d\left(\lambda_j\right) + \left.\frac{\partial d}{\partial s}\right|_{s=\lambda_j} \frac{\partial \lambda_j}{\partial g}\delta g + \left(\mathbf{g} + \delta g \times \mathbf{e}_i + \lambda_j \mathbf{f} + \frac{\partial \lambda_j}{\partial g}\delta g\,\mathbf{f}\right)^T \left(\mathbf{N}\left(\lambda_j\right) + \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_j} \frac{\partial \lambda_j}{\partial g}\delta g\right)\mathbf{b} = 0 \tag{9.20}$$

Combining equations (9.16) and (9.19) and neglecting δg^2 and higher orders of smallness,

$$\left.\frac{\partial d}{\partial s}\right|_{s=\lambda_j}\left(\frac{\partial \lambda_j}{\partial g}\right) + \left(\mathbf{g} + \lambda_j \mathbf{f}\right)^T \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_j}\left(\frac{\partial \lambda_j}{\partial g}\right)\mathbf{b} + \left(\mathbf{e}_i + \frac{\partial \lambda_j}{\partial g}\mathbf{f}\right)^T \mathbf{N}\left(\lambda_j\right)\mathbf{b} = 0 \tag{9.21}$$

The sensitivities of the poles are then given by,

$$S_{ji}^{g} = \frac{\partial \lambda_j}{\partial g_i} = \frac{-\mathbf{e}_i^T \mathbf{N}\left(\lambda_j\right)\mathbf{b}}{\left.\frac{\partial d}{\partial s}\right|_{s=\lambda_j} + \left(\mathbf{g} + \lambda_j \mathbf{f}\right)^T \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_j}\mathbf{b} + \mathbf{f}^T \mathbf{N}\left(\lambda_j\right)\mathbf{b}} \tag{9.22}$$

Similar expressions are easily obtained for the sensitivities of the poles with respect to the velocity feedback gains $\mathbf{f}$,

$$S_{ji}^{f}=\frac{-\lambda_j\mathbf{e}_i^T\mathbf{N}(\lambda_j)\mathbf{b}}{\left.\frac{\partial d}{\partial s}\right|_{s=\lambda_j}+(\mathbf{g}+\lambda_j\mathbf{f})^T\left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_j}\mathbf{b}+\mathbf{f}^T\mathbf{N}(\lambda_j)\mathbf{b}} \tag{9.23}$$

And the sensitivities of the open-loop eigenvalue σ_j (when $\mathbf{g},\mathbf{f}=\mathbf{0}$) are then given by,

$$S_{ji}^{g}=\frac{-\mathbf{e}_i^T\mathbf{N}(\sigma_j)\mathbf{b}}{\left(\left.\frac{\partial d}{\partial s}\right|_{s=\sigma_j}\right)};\quad S_{ji}^{f}=\frac{-\sigma_j\mathbf{e}_i^T\mathbf{N}(\sigma_j)\mathbf{b}}{\left(\left.\frac{\partial d}{\partial s}\right|_{s=\sigma_j}\right)} \qquad (9.24), (9.25)$$

In practice the terms $\frac{\partial d}{\partial s}$ and $\frac{\partial \mathbf{N}}{\partial s}$ may be obtained after curve fitting $d(\mathrm{i}\omega)$ and $\mathbf{N}(\mathrm{i}\omega)$ to measured receptances using rational fraction polynomials, as was done by Mottershead *et al.*[M16].

9.3 Open-loop sensitivity from the eigenvectors

From equations (9.12) and (9.13) we see that,

$$d(s)=\prod_{k=1}^{2n}(s-\sigma_k) \tag{9.26}$$

and, using the chain rule,

$$\frac{\partial d(s)}{\partial s}=\sum_{k=1}^{2n}\frac{d(s)}{(s-\sigma_k)};\quad \left.\frac{\partial d(s)}{\partial s}\right|_{s=\sigma_j}=\sum_{k=1}^{2n}\frac{d(\sigma_j)}{(\sigma_j-\sigma_k)} \qquad (9.27), (9.28)$$

Also, from equations (9.12) and (9.13),

$$\mathbf{N}(s)=\sum_{k=1}^{2n}\frac{\mathbf{v}_k\mathbf{v}_k^T}{(s-\sigma_k)}d(s);\quad \mathbf{N}(\sigma_j)=\sum_{k=1}^{2n}\frac{\mathbf{v}_k\mathbf{v}_k^T}{(\sigma_j-\sigma_k)}d(\sigma_j) \qquad (9.29), (9.30)$$

And by transposition of scalar products,

$$-\mathbf{e}_i^T \mathbf{v}_j \mathbf{v}_j^T \mathbf{b} = -\mathbf{v}_j^T \mathbf{b} \mathbf{e}_i^T \mathbf{v}_j \tag{9.31}$$

Then, by combining equations (9.24), (9.26), (9.28), (9.30) and (9.31), the following equation is revealed,

$$S_{ji}^g = -\mathbf{v}_j^T \mathbf{b} \mathbf{e}_i^T \mathbf{v}_j \tag{9.32}$$

This equation is found straightforwardly from the well-known expression [R8], [N3],

$$\frac{\partial \sigma_j}{\partial g} = \left(\sigma_j \mathbf{v}_j^T \;\; \mathbf{v}_j^T\right)^T \frac{\partial}{\partial g}\left(\sigma_j \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} + \mathbf{b}\mathbf{e}_i^T f \end{bmatrix} + \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} + \mathbf{b}\mathbf{e}_i^T g \end{bmatrix}\right)\begin{pmatrix} \mathbf{v}_j \sigma_j \\ \mathbf{v}_j \end{pmatrix} \tag{9.33}$$

which reduces to,

$$S_{ji}^g = -\mathbf{v}_j^T \frac{\partial}{\partial g}\left[\mathbf{K} + \mathbf{b}\mathbf{e}_i^T g\right]\mathbf{v}_j \tag{9.34}$$

Likewise,

$$S_{ji}^f = -\sigma_j \mathbf{v}_j^T \mathbf{b} \mathbf{e}_i^T \mathbf{v}_j \tag{9.35}$$

The eigenvectors in equations (9.32) and (9.35) are the normalised eigenvectors from equation (9.7), which are generally only available from a known **M**, **C**, **K** model and not from experimental tests.

9.4 Assignment of eigenvalue sensitivities

By rearranging equation (9.22) and assigning $S_{ji}^{g} = \alpha$,

$$\mathbf{b}^T \frac{\partial \mathbf{N}}{\partial s}\bigg|_{s=\lambda_j} \mathbf{g} + \left(\lambda_j \mathbf{b}^T \frac{\partial \mathbf{N}}{\partial s}\bigg|_{s=\lambda_j} + \mathbf{b}^T \mathbf{N}(\lambda_j) \right) \mathbf{f} = -\frac{\mathbf{e}_i^T \mathbf{N}(\lambda_j)\mathbf{b}}{\alpha} - \frac{\partial d}{\partial s}\bigg|_{s=\lambda_j} \tag{9.36}$$

and similarly for equation (9.23), assigning $S_{ji}^{f} = \beta$,

$$\mathbf{b}^T \frac{\partial \mathbf{N}}{\partial s}\bigg|_{s=\lambda_j} \mathbf{g} + \left(\lambda_j \mathbf{b}^T \frac{\partial \mathbf{N}}{\partial s}\bigg|_{s=\lambda_j} + \mathbf{b}^T \mathbf{N}(\lambda_j) \right) \mathbf{f} = -\frac{\lambda_j \mathbf{e}_i^T \mathbf{N}(\lambda_j)\mathbf{b}}{\beta} - \frac{\partial d}{\partial s}\bigg|_{s=\lambda_j} \tag{9.37}$$

linear equations in **g** and **f** are obtained that may be added to characteristic equations for the pole placement, thereby allowing the assignment of both poles and the sensitivities of the poles.

9.5 Numerical examples

Example 9.1.

We consider the 3 degree of freedom system defined by the matrices,

$$\mathrm{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$

The open-loop poles are found to be,

$\sigma_{1,2} = -0.0165 \pm 0.5516\mathrm{i}$

$\sigma_{3,4} = -0.1889 \pm 1.6044\mathrm{i}$

$\sigma_{5,6} = -0.2527 \pm 2.2288\mathrm{i}$

It is assumed that sensors and actuators are available at each of the three masses. The actuator selection matrix is assumed to be $\mathbf{b} = [1 \quad 1 \quad 1 \quad 1]^T$.

We wish to assign the first two pairs of poles and the sensitivity of the third pair of poles with respect to the first term in $\mathbf{g}$ as follows,

$$\lambda_{1,2} = -0.02 \pm 0.8\text{i}$$
$$\lambda_{3,4} = -0.3 \pm 1.9\text{i}$$
$$\alpha_{51}^{g} = -0.6 - 0.0001\text{i}$$
$$\alpha_{61}^{g} = -0.6 + 0.0001\text{i}$$

The open-loop sensitivities for the third pair of pole are:

$$S_{51}^{g} = -0.00119 - 0.03162\text{i}$$
$$S_{61}^{g} = -0.00119 + 0.03162\text{i}$$

Solving for the control gains,

$$\begin{bmatrix} \mathbf{r}_1{}^T & \lambda_1 \mathbf{r}_1{}^T \\ \mathbf{r}_2{}^T & \lambda_2 \mathbf{r}_2{}^T \\ \mathbf{r}_3{}^T & \lambda_3 \mathbf{r}_3{}^T \\ \mathbf{r}_4{}^T & \lambda_4 \mathbf{r}_4{}^T \\ \mathbf{a}_5{}^T & \mathbf{c}_5{}^T \\ \mathbf{a}_6{}^T & \mathbf{c}_6{}^T \end{bmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ y_5 \\ y_6 \end{pmatrix}$$

with $\mathbf{a}_j = \left.\dfrac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_j} \mathbf{b}$

$$\mathbf{c}_j = \lambda_j \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_j} \mathbf{b} + \mathbf{N}(\lambda_j)\mathbf{b}$$

$$y_j = -\frac{\mathbf{e}_i^T \mathbf{N}(\lambda_j)\mathbf{b}}{\alpha_{ij}^g} - \left.\frac{\partial d}{\partial s}\right|_{s=\lambda_j}$$

leads to,

$$\mathbf{g} = \begin{pmatrix} 1.6425 \\ 2.1328 \\ -1.4327 \end{pmatrix} \qquad \mathbf{f} = \begin{pmatrix} 1.0143 \\ -0.2658 \\ -0.2515 \end{pmatrix}$$

The closed loop poles are found to be,

$$\lambda_{1,2} = -0.02 \pm 0.8\mathrm{i}$$
$$\lambda_{3,4} = -0.3 \pm 1.9\mathrm{i}$$
$$\lambda_{5,6} = -0.217 \pm 2.4560\mathrm{i}$$

and, as expected, the sensitivities of the third pair of poles are,

$$S_{51}^g = -0.6 - 0.0001\mathrm{i}$$
$$S_{61}^g = -0.6 + 0.0001\mathrm{i}$$

Example 9.2.

The 3 dof system in in the previous example is considered. It is required to assign the first pair of poles. Also, the sensitivity of the first pair of poles is to be 100 times greater than the sensitivity of the second pair. The sensitivity of the third pair of poles is to take a prescribed value. All the sensitivities are with respect to the first term in $\mathbf{g}$.

$$\lambda_{1,2} = -0.02 \pm 0.8\mathrm{i}$$
$$\alpha_{11}^g = 100\alpha_{21}^g$$
$$\alpha_{31}^g = 100\alpha_{41}^g$$
$$\alpha_{51}^g = -0.3 - 0.0005\mathrm{i}$$
$$\alpha_{61}^g = -0.3 + 0.0005\mathrm{i}$$

In this case we need to solve,

$$\begin{bmatrix} \mathbf{r}_1^T & \lambda_1 \mathbf{r}_1^T \\ \mathbf{r}_2^T & \lambda_2 \mathbf{r}_2^T \\ \mathbf{p}_3^T & \mathbf{q}_3^T \\ \mathbf{p}_4^T & \mathbf{q}_4^T \\ \mathbf{a}_5^T & \mathbf{c}_5^T \\ \mathbf{a}_6^T & \mathbf{c}_6^T \end{bmatrix} \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \nu_3 \\ \nu_4 \\ y_5 \\ y_6 \end{pmatrix}$$

where

$$\mathbf{p}_3 = -\left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_1} \mathbf{b} + \gamma_3 \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_3} \mathbf{b}$$

$$\mathbf{p}_4 = -\left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_2} \mathbf{b} + \gamma_4 \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_4} \mathbf{b}$$

$$\mathbf{q}_3 = -\left(\lambda_1 \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_1} \mathbf{b} + \mathbf{N}(\lambda_1)\mathbf{b} \right) + \gamma_3 \left(\lambda_3 \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_3} \mathbf{b} + \mathbf{N}(\lambda_3)\mathbf{b} \right)$$

$$\mathbf{q}_4 = -\left(\lambda_2 \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_2} \mathbf{b} + \mathbf{N}(\lambda_2)\mathbf{b} \right) + \gamma_4 \left(\lambda_4 \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\lambda_4} \mathbf{b} + \mathbf{N}(\lambda_4)\mathbf{b} \right)$$

$$\nu_3 = \left.\frac{\partial d}{\partial s}\right|_{s=\lambda_1} - \gamma_3 \left.\frac{\partial d}{\partial s}\right|_{s=\lambda_3}$$

$$\nu_4 = \left.\frac{\partial d}{\partial s}\right|_{s=\lambda_2} - \gamma_4 \left.\frac{\partial d}{\partial s}\right|_{s=\lambda_4}$$

and

$$\gamma_3 = \frac{100\left(\mathbf{e}_1^T \mathbf{N}(\lambda_3)\mathbf{b}\right)}{\left(\mathbf{e}_1^T \mathbf{N}(\lambda_1)\mathbf{b}\right)}$$

$$\gamma_4 = \frac{100\left(\mathbf{e}_1^T \mathbf{N}(\lambda_4)\mathbf{b}\right)}{\left(\mathbf{e}_1^T \mathbf{N}(\lambda_2)\mathbf{b}\right)}$$

The control gains are determined as,

$$\mathbf{g} = \begin{pmatrix} -12.2622 \\ 6.5184 \\ 0.8303 \end{pmatrix} \qquad \mathbf{f} = \begin{pmatrix} -0.4372 \\ 0.0773 \\ -0.3759 \end{pmatrix}$$

The closed-loop poles are found to be,

$$\lambda_{1,2} = -0.02 \pm 0.8\mathrm{i}$$
$$\lambda_{3,4} = -0.0237 \pm 0.8238\mathrm{i}$$
$$\lambda_{5,6} = -0.2813 \pm 2.669\mathrm{i}$$

and the closed-loop sensitivities are returned with the demanded values as,

$$S_{51}^{g} = -0.3 - 0.0005\mathrm{i}$$
$$S_{61}^{g} = -0.3 + 0.0005\mathrm{i}$$
$$S_{11}^{g} = -0.1757 + 4.586\mathrm{i}$$
$$S_{21}^{g} = -0.1757 - 4.586\mathrm{i}$$
$$S_{31}^{g} = -0.001757 + 0.04586\mathrm{i}$$
$$S_{41}^{g} = -0.001757 - 0.04586\mathrm{i}$$

9.6 Conclusion

In this chapter the sensitivities of the poles with respect to the control gains are derived from the matrix of measured receptances. Single-input state feedback is considered, resulting in a unit rank modification to the stiffness and damping matrices of the closed-loop system. The eigenvalue sensitivities are assigned so that selected eigenvalues are made sensitive, while other eigenvalues are rendered insensitive. The sensitivity equations are linear in the unknown control gains allowing us to simultaneously assign poles and their sensitivities. Numerical examples demonstrate the feasibility of the method.

Chapter 10

CONCLUSION AND FUTURE WORK

10.0 Conclusion

In this research structural modification using passive modification and active vibration control by the receptance method is presented. The theory of the receptance method for the inverse problem of pole and zero assignment using passive modification is well known [Chapters 2, 3]. However it was used for the first time by Ram and Mottershead [R6] in active vibration control and is developed further in this thesis.

The structural modification theory based on the receptance method was applied to a Lynx Mark 7 helicopter tail-cone [M15]. Physical modifications such as large overhanging masses require the measurement of rotational receptances at the connection points. The use of an X-block attachment proposed [Chapter 4] enables us to measure the full receptance matrix using the multi-input-multi-output estimator. The flexibility of the X-block as well as the measured spectral densities was included in the formulation of H_1 and H_2 estimators to determine the receptance matrix at the connection point of the X-block to the tail-cone. It was demonstrated that measuring the rotational receptances are very difficult and require high levels of specialist expertise.

The purpose of eigenvalue assignment in structural dynamics is to suppress vibration. Vibration suppression can be achieved for example by moving poles of the system further to the left-hand side of the complex plane; assigning a zero at the tuned frequency of a classical vibration absorber; and moving the closed-loop poles away from the resonance frequencies.

The inverse eigenvalue problem for vibration absorption using passive and active control was discussed [M14]. The main advantage of passive modification over active vibration control is that the system is guaranteed to be stable. However, there are significant disadvantages such as, 1) the form of the modification that can be realised in practice (symmetry, positive-definiteness, reciprocity, bandedness of the matrix) is restrictive, 2) rotational receptances are very difficult to measure and require high levels of specialist expertise [Chapter 4], and 3) the rank of the modification should be at least equal to the number of eigenvalues to be assigned. These disadvantages do not apply to eigenvalue assignment by active control. The main issue in active vibration control is the stability of the closed-loop system.

A new approach based upon measured receptances was described [R6], [M16] for the assignment of poles and zeros by active vibration control. The significant advantage of the receptance method over the conventional state-space method is the use of measured frequency response function $\mathbf{H}(\mathrm{i}\omega)$ instead of the system parameters **M**, **C**, **K** typically obtained from finite element models so that there would be no need for the evaluation of the system matrices **M**, **C**, **K**. Using this method one can control the vibration modes with measuring just a small number of states. Therefore, there is no need for model reduction or the estimation of the unmeasured states using an observer.

In practice, FE models introduce unnecessarily complications to the controller design [Chapter 5]. Firstly, there is not an equivalent approach that can deal with different form of damping that can be found in real structures. FE models normally neglect damping or assume *ad hoc* Rayleigh (proportional) damping. In active control the model of damping is very important in the eigenvalue assignment problem. The lack of damping can lead to an inaccurate feedback controller design and may cause the closed-loop eigenvalues move to the instability region of the complex plane. Secondly, FE models used in design can be very large therefore computationally expensive and require model reduction, truncation or other approximations which can degrade the performance of the controller. Finally FE models contain many assumptions and approximations, therefore the controller should be insensitive to the ill-defined FE parameters such as joints and boundary conditions. These unnecessary complications can be avoided by the adoption of the receptance method.

In the state feedback control the method leads to linear characteristic equations in the unknown gains for the assignment of poles and zeros. The closed-loop dynamic stiffness is changed by the rank-1 modification as a consequence of the single input state feedback control [Chapter 6]. Although the state feedback theory is simplest because of the linear equations in the unknown control gains, the output feedback control allows the use of collocated actuators and sensors in multiple-input-multiple-output (MIMO) systems [Chapter 7]. The transfer function of a lightly damped system with a collocated actuator-sensor pair displays a line of interlacing poles and zeros just to the left of the imaginary axis and therefore offers robustly stable solutions.

Experiments were carried out on the T-shaped plate [Chapter 8] using two sets of collocated sensors and inertial actuators. The assignment of two pairs of complex conjugate poles corresponding to the first two modes, assignment of zeros and simultaneous assignment of poles and zeros were considered. The open-loop receptance $\mathbf{H}(s)$ was determined from the measured $\mathbf{H}(\mathrm{i}\omega)$ and a rational fraction polynomial was fitted to represent the transfer function of the receptance. In addition to velocity feedback, for active damping, the method uses displacement feedback, for active stiffness, thereby enabling the assignment of both poles and zeros to desired locations in the complex s-plane. Control gains were obtained by solving the nonlinear characteristic equations using the Newton iterative method. After application of the active control method, the peaks of the closed-loop-system receptances were in agreement with the imaginary parts of the assigned poles. The stability robustness was improved by applying a constraint to the singular values of the matrix return difference resulting in higher damping in the closed-loop response.

It is often desirable not only to place eigenvalues at chosen locations in the complex plane, but also to make them sensitive or insensitive with respect to the feedback control gains. The controller can be designed to reduce the control effort by increasing the sensitivities of the assigned eigenvalues while minimizing the sensitivities of other eigenvalues to avoid instability caused by spillover. A new approach based on the receptance method was developed [Chapter 9] to design a controller, which renders selected eigenvalues sensitive while others are made insensitive.

The receptance method offers immense theoretical development and practical applications. The method is generic, having wide applications to many industries including automotive, civil engineering and aerospace. It is not limited to particular types of structures or by physical size or complexity and does not rely on mathematical models which may contain inaccuracies or assumptions.

10.1 Future work

The receptance method has potential applications in many industries and can be developed further into many directions of research. The basic theories for pole-zero assignment has been developed and presented in this thesis.

Other aspects that can be considered as future work may be summarized here:

1. A combination of passive and active control based on the theory of receptance method is of particular interest. Both advantages of passive and active control can be exploited. An example of this might be the introduction of passive constrained layer damping to active elements [S7].

2. Inertial actuators were used in this research for the experimental work. Different types of actuators such as piezoelectric actuators and their operating characteristics can be considered and the performance of the system for each type can be analyzed.

3. Non-collocated actuators and sensors can be considered for the pole assignment problem. The closed-loop system may be extremely sensitive to system parameters in this case and, therefore requires sophisticated techniques to achieve robust control.

4. Experiments for the pole assignment may be carried out on real systems such as helicopter fuselage.

5. An adaptive controller may be designed such that the response of the system under different environmental or load conditions remains unchanged. For example, the dynamic behavior of a helicopter fuselage may change when carrying heavy equipment or different fuel levels. An objective may be defined to maintain certain natural frequencies invariant under such time-varying load conditions.

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